NINTH ANNUAL UW MADISON
UNDERGRADUATE MATH COMPETITION

1. Suppose f is a differentiable function. Show that there exists α between 0 and 2π such that the vector $(f(\alpha), f'(\alpha)+1)$ is perpendicular to the unit vector $(\cos(\alpha), \sin(\alpha))$.

2. Find all pairs of integers $a, p > 1$ such that p is prime and $\log_a(a+p)$ is rational (where $\log_a$ is the logarithm base a).

3. Let A be a 100×100 matrix such that each column of A contains at most two entries 1, and the rest of entries are zero. What is the maximal value of $\det(A)$?

4. In each square of a 2025×2025 board there is a light and a switch. Flicking a switch changes the state of the light (on to off and off to on) in its square, as well as all the lights in the same row and in the same column. In the beginning, all lights are off. How many different configurations of lights can be obtained by using the switches?

5. Let $f(x) > 0$ be a function defined for all x and suppose M is a constant such that $f''(x) \leq M$ for all x (in particular, $f(x)$ is two times differentiable.) Show that

$$(\sqrt{f})' \leq \sqrt{M/2}$$

6. Let n be an integer. Consider a random sequence of sets $B_0, B_1, \ldots, B_n$ chosen recursively so that $B_0 = \{1, \ldots, 3^n\}$, and B_{k+1} is a randomly chosen subset of B_k for $k \geq 0$, with all choices equally likely. Denote by P_n the probability that $B_n = \emptyset$. Find

$$\lim_{n\to\infty} P_n.$$

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