UW–Madison Putnam Club

April 16, 2025 — Differential Equations

1. Suppose $h\colon \mathbb{R}^2 \to \mathbb{R}$ has continuous partial derivatives and satisfies $a\partial_x h + b\partial_y h = h$ for some $a, b \in \mathbb{R}$. Prove that if h is bounded, it is identically zero. [Putnam 2010 A3]

2. Let $u \coloneqq \mathbb{R}^n \to [0, \infty)$ be twice differentiable and satisfy $\Delta u = 0$. Show that u is constant.

3. Show that there is no strictly increasing function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f' = f \circ f$. [Putnam 2010 B5]

4. Let $x_1, \ldots, x_n\colon \mathbb{R} \to \mathbb{R}$ be differentiable functions satisfying

$$\begin{aligned} x_1' &= a_{11}x_1 + \ldots + a_{1n}x_n, \\ x_2' &= a_{21}x_1 + \ldots + a_{2n}x_n, \\ &\vdots \\ x_n' &= a_{n1}x_1 + \ldots + a_{nn}a_n \end{aligned}$$

for some constants $a_{ij} \geq 0$. Suppose $x_i(t) \to 0$ as $t \to \infty$ for all i. Are the functions $x_1, \ldots, x_n$ necessarily linearly dependent? [Putnam 1995 A5]

5. Suppose a solution u of $\partial_t u + u\partial_x u = \partial_x^2 u$ has the form $u(t, x) = h(x - vt)$ for some smooth $h\colon \mathbb{R} \to \mathbb{R}$ and $v \in \mathbb{R}$. Assume the limits $h(-\infty) = a$ and $h(+\infty) = b$ exist and all derivatives of h decay to zero at infinity. Show that $v = \frac{a+b}{2}$.

6. Let $f(x) = \mathrm{e}^{x^2}$. Find g such that a "calculus first-year's dream" is true: $(fg)' = f'g'$. [Putnam 1988 A2]

7. Find all differentiable functions $f\colon (0, \infty) \to (0, \infty)$ for which there is a positive real number a such that

$$f'\left(\frac{a}{x}\right) = \frac{x}{f(x)}$$

for all $x > 0$. [Putnam 2009 B5]

Hints

2. If $\Delta u = 0$, it satisfies the "mean value property": $u(x) = \frac{1}{\operatorname{vol} B_r} \int_{B_r(x)} u(y)\, \mathrm{d}y$ for all $x \in \mathbb{R}^n$ and all $r > 0$. That is, $u(x)$ is equal to the average of u over any ball centered at x.

3. Play around with f^{-1}.

7. Differentiate and try to eliminate evaluations at a/x.