ALGEBRA QUALIFYING EXAM, AUGUST 2014

In all cases, when an example is requested, you should both provide the example and a proof that the object you write down actually is an example.

(1) For $n \geqslant 1$ consider the ring $R = M_n(\mathbb{Z}/4\mathbb{Z})$ of $n \times n$ matrices with entries in the ring $\mathbb{Z}/4\mathbb{Z}$. It is naturally a $\mathbb{Z}/4\mathbb{Z}$-algebra.

- (a) Prove that $R \otimes_{\mathbb{Z}/4\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$ is isomorphic to $M_n(\mathbb{Z}/2\mathbb{Z})$. (Make sure to provide a careful description of the isomorphism you construct.) Use general properties of tensor products to argue that the natural map
$$M_n(\mathbb{Z}/4\mathbb{Z}) = R \to R \otimes_{\mathbb{Z}/4\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} = M_n(\mathbb{Z}/2\mathbb{Z})$$
is surjective.
- (b) Describe the kernel K of the above surjective map
$$M_n(\mathbb{Z}/4\mathbb{Z}) \to M_n(\mathbb{Z}/2\mathbb{Z}).$$
- (c) Find all two-sided ideals of the ring R. (Hint: Use the computations above.)

(2) Let G be a finite group and A a subgroup of $\mathrm{Aut}(G)$.

- (a) Suppose G is the cyclic group $\mathbb{Z}/6\mathbb{Z}$ and A is the full automorphism group $\mathrm{Aut}(G)$. What are the orbits of the action of A on G?
- (b) Let G be a non-trivial finite group. Show that two elements in the same orbit of A on G must have the same order.
- (c) Show that for any non-trivial finite group G there are always at least two orbits of A on G. Prove that there are exactly two orbits for some A if and only if G is an elementary abelian p-group for some prime p.

(3) This problem concerns eigenvectors of linear transformations.

- (a) Let $V \neq 0$ be a finite-dimensional vector space over $\mathbb{C}$ and let $T : V \to V$ be a linear transformation. Prove that T has an eigenvector.
- (b) Give an example of a finite-dimensional vector space $V \neq 0$ over $\mathbb{R}$ and a linear transformation $T : V \to V$ which does not have an eigenvector.
- (c) Does a linear transformation of an *infinite-dimensional* vector space over $\mathbb{C}$ have to have an eigenvector? Either prove this is the case, or give an example of a linear transformation of an infinite-dimensional vector space which has no eigenvector.
- (d) Suppose that T and U are two linear transformations of a finite-dimensional vector space V over $\mathbb{C}$ which commute with each other. Prove that there is some $v \in V$ which is an eigenvector for both T and U.

(4) Let R be a commutative ring and M an R-module. Recall that a prime ideal P is an associated prime of M if there exists some $m \in M$ such that P consists of those f such that $fm = 0$; i.e., P is the *annihilator* of some $m \in M$.

(a) Let

$$0 \to M' \to M \to M'' \to 0$$

be an exact sequence of R-modules, and let P be an associated prime of M. Prove that P is an associated prime of either M' or M''.

(b) Is the converse true? That is, if P is an associated prime of either M' or M'', must it be the case that P is an associated prime of M?

(5) It is well-known that if H_1 and H_2 are two subgroups of a group G, then the index $[G : H_1 \cap H_2]$ is at most $[G : H_1][G : H_2]$. But the analogous statement for field extensions is not true.

(a) Let K be the splitting field of $\mathbb{Q}(2^{1/3})$ over $\mathbb{Q}$. Give an example of two subfields L_1 and L_2 of K such that K/L_1 and K/L_2 are both quadratic extensions, but $K/(L_1 \cap L_2)$ has degree greater than 4.

(b) Let K be the field of rational functions $\mathbb{C}(x)$. Give an example of two subfields L_1 and L_2 of K such that K/L_1 and K/L_2 are both quadratic extensions and such that $K/(L_1 \cap L_2)$ has *infinite* degree.