ALGEBRA QUALIFYING EXAM, AUGUST 2017

1. For this problem and this problem only your answer will be graded on correctness alone, and no justification is necessary.

Consider the ring $\mathbb{C}[x]$ and its subrings $\mathbb{C} \subset \mathbb{C}[x]$ and $\mathbb{C}[x^2] \subset \mathbb{C}[x]$. Given any two $\mathbb{C}[x]$-modules M and N, we can consider their tensor product over any of the three rings:

$$M \otimes_{\mathbb{C}[x]} N, \quad M \otimes_{\mathbb{C}} N, \quad \text{and} \quad M \otimes_{\mathbb{C}[x^2]} N.$$

The tensor products are modules over the corresponding rings, and, in particular, all three are vector spaces over $\mathbb{C}$.

Put $M = \mathbb{C}[x]/(x^2 + x)$ and $N = \mathbb{C}[x]/(x - 1)$.

(a) What is the dimension of $M \otimes_{\mathbb{C}[x]} N$ as a vector space over $\mathbb{C}$?
(b) What is the dimension of $M \otimes_{\mathbb{C}} N$ as a vector space over $\mathbb{C}$?
(c) What is the dimension of $M \otimes_{\mathbb{C}[x^2]} N$ as a vector space over $\mathbb{C}$?

2. Let K be a field, and let A be an $n \times n$-matrix over K. Suppose that $f \in K[x]$ is an *irreducible* polynomial such that $f(A) = 0$. Show that $\deg(f)|n$.

3. What is the smallest n such that the 3-Sylow subgroup of S_n is non-abelian? (You may use the Sylow theorem that all Sylow subgroups are conjugate, so that one 3-Sylow subgroup is non-abelian if and only if they all are.)

4. Suppose that $K \subset \mathbb{C}$ is a Galois extension of $\mathbb{Q}$, $[K : \mathbb{Q}] = 4$, and that $\sqrt{-m} \in K$ for some positive integer m. Show that

$$Gal(K/\mathbb{Q}) \simeq (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}).$$

5. The Noether normalization lemma implies that the ring $B = \mathbb{Q}[x, y]/(xy)$ can be realized as a finite extension of $\mathbb{Q}[t]$; that is, B is a finitely generated $\mathbb{Q}[t]$-module.

(a) Consider the ring homomorphism $\mathbb{Q}[t] \to B$ sending t to x. Show that B is *not* a finite extension of $\mathbb{Q}[t]$.
(b) Write down an explicit map $\mathbb{Q}[t] \to B$ that turns B into a finite extension of $\mathbb{Q}[t]$ and prove that the extension is indeed finite.
(c) Consider B as a $\mathbb{Q}[t]$-module via the map you constructed in the previous question. Is B a flat $\mathbb{Q}[t]$-module? Justify your answer.