ALGEBRA QUALIFYING EXAM, AUGUST 2017

Final draft 08/10

1. For this problem and this problem only your answer will be graded on correctness alone, and no justification is necessary.

Consider the ring $\mathbb{C}[x]$ and its subrings $\mathbb{C} \subset \mathbb{C}[x]$ and $\mathbb{C}[x^2] \subset \mathbb{C}[x]$. Given any two $\mathbb{C}[x]$-modules M and N, we can consider their tensor product over any of the three rings:

$$M \otimes_{\mathbb{C}[x]} N, \quad M \otimes_{\mathbb{C}} N, \quad \text{and} \quad M \otimes_{\mathbb{C}[x^2]} N.$$

The tensor products are modules over the corresponding rings, and, in particular, all three are vector spaces over $\mathbb{C}$.

Put $M = \mathbb{C}[x]/(x^2 + x)$ and $N = \mathbb{C}[x]/(x - 1)$.

(a) What is the dimension of $M \otimes_{\mathbb{C}[x]} N$ as a vector space over $\mathbb{C}$?
(b) What is the dimension of $M \otimes_{\mathbb{C}} N$ as a vector space over $\mathbb{C}$?
(c) What is the dimension of $M \otimes_{\mathbb{C}[x^2]} N$ as a vector space over $\mathbb{C}$?

2. Let K be a field, and let A be an $n \times n$-matrix over K. Suppose that $f \in K[x]$ is an *irreducible* polynomial such that $f(A) = 0$. Show that $\deg(f)|n$.

3. What is the smallest n such that the 3-Sylow subgroup of S_n is non-abelian? (You may use the Sylow theorem that all Sylow subgroups are conjugate, so that one 3-Sylow subgroup is non-abelian if and only if they all are.)

4. Suppose that $K \subset \mathbb{C}$ is a Galois extension of $\mathbb{Q}$, $[K : \mathbb{Q}] = 4$, and that $\sqrt{-m} \in K$ for some positive integer m. Show that

$$Gal(K/\mathbb{Q}) \simeq (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}).$$

5. The Noether normalization lemma implies that the ring $B = \mathbb{Q}[x, y]/(xy)$ can be realized as a finite extension of $\mathbb{Q}[t]$; that is, B is a finitely generated $\mathbb{Q}[t]$-module.

(a) Consider the ring homomorphism $\mathbb{Q}[t] \to B$ sending t to x. Show that B is *not* a finite extension of $\mathbb{Q}[t]$.
(b) Write down an explicit map $\mathbb{Q}[t] \to B$ that turns B into a finite extension of $\mathbb{Q}[t]$ and prove that the extension is indeed finite.
(c) Consider B as a $\mathbb{Q}[t]$-module via the map you constructed in the previous question. Is B a flat $\mathbb{Q}[t]$-module? Justify your answer.

Solutions

1. The answers are 0, 2, and 1, respectively.

2. Let g be the characteristic polynomial of A. Then $\deg(g) = n$. Since every root of g is an eigenvalue of A, it must be a root of f as well. This implies that f is the only irreducible factor of g, and therefore f is (up to scaling) a power of f.

3. $n = 9$. Indeed, for $n < 9$, the maximal power of 3 dividing $n!$ is at most 9, and groups of order 3 and 9 are abelian. Looking at S_9, we can describe a Sylow subgroup explicitly (one of them is generated by (123) and (147)(258)(369)).

4. Since $\mathbb{Q}(\sqrt{-m}) \subset K$, it corresponds to a two-element subgroup in the Galois group $G = Gal(K/Q)$. Therefore, there is a non-trivial involution $f \in G$ keeping $\mathbb{Q}(\sqrt{-m})$ fixed. However, the complex conjugation is another involution in G. This rules out the case $G \simeq \mathbb{Z}/4\mathbb{Z}$, and therefore $G \simeq (\mathbb{Z}/2\mathbb{Z})^2$, as claimed.

5. For (a), we note that the $\mathbb{Q}[t]$-submodules generated by y^k form an ascending chain that does not stabilize, therefore, B is not finitely generated (there are also more direct ways to solve this).

For (b) and (c), put $u = (x+y)/2$, $t = (x-y)/2$. Then $B = \mathbb{Q}[u,t]/(u^2 - t^2 - 1)$. Since $u^2 - t^2 - 1$ is a monic polynomial of u, we see that 1 and u form a basis of B as a $\mathbb{Q}[t]$-module. In particular, it is flat.