ALGEBRA QUALIFYING EXAM, AUGUST 2018

1.

(a) Give an example of a ring R and a short exact sequence of R-modules $0 \to A \to B \to C \to 0$ that is not split exact.
(b) Give an example of a flat $\mathbb{Z}$-module that is not free.
(c) Describe all zero-divisors and all units in $\mathbb{Q}[x]/(x^2 - 1)$.

2. For a finite group G, denote by $s(G)$ the number of its subgroups.

(a) Show that $s(G)$ is finite.
(b) Show that if H is a nontrivial normal subgroup of G, then $s(G/H) < s(G)$.
(c) Show that $s(G) = 2$ if and only if G is cyclic of prime order.
(d) Show that $s(G) = 3$ if and only if G is a cyclic group whose order is a square of a prime.

3. Denote by $M_n(\mathbb{R})$ the ring of all $n \times n$ matrices over $\mathbb{R}$.

(a) Show that for any $A \in M_n(\mathbb{R})$, there exists $B \in M_n(\mathbb{R})$ such that $AB = 0$ and $\mathrm{rk}(A) + \mathrm{rk}(B) = n$.
(b) Prove or disprove that for any $A \in M_n(\mathbb{R})$, there exists $B \in M_n(\mathbb{R})$ such that $AB = BA = 0$ and $\mathrm{rk}(A) + \mathrm{rk}(B) = n$.

4.

(a) Give an example of a field extension $K/\mathbb{Q}$ whose Galois group is $\mathbb{Z}/4\mathbb{Z}$, and prove that it is such an example.
(b) Let K be the field $\mathbb{F}_q(t)$ and let $L = \mathbb{F}_q(t^{1/p})$. The extension L/K is inseparable, thus not Galois. Explain why there are no nontrivial field automorphisms of L fixing K.

5. Let A be a two-dimensional (unital) algebra over a field F. This means that A is an associative, but not necessarily commutative, ring with a unit that contains F as a subring such that the elements of F commute with all alements of A (that is, F is in the center) and A is two-dimensional as a vector space over F.

(a) Show that A must in fact be commutative.
(b) Show that if F is algebraically closed, then either $A \simeq F \times F$ or $A \simeq F[x]/(x^2)$.
(c) Suppose $F = \mathbb{R}$. List (with a proof) all possibilities for A, up to isomorphism.