6/5/2025

Intro to Bun_G

Logistics: a) Next Thursday Def Theory. – Dima 2PM-3PM.
After – Wed 2PM–3PM Bun_G
Thur 2PM – 3PM Def Theory } Weekly.

b) Bun_G – Sorger's "Moduli of Princ. G-Bundles"
– Heinloth "Moduli Stack of V.b on Curves"

c) Recorded Talks – you can object.

d) Sign up for talk plz!

Generalities on (princ) G-bundles (G smooth linear alg group / $\mathbb{F}$).
X-scheme

Def: A prin. G-bundle on $X_{ét}$ (scheme w/ étale site) is

a) a scheme $E \xrightarrow{\pi} X$ w/ $E \circlearrowleft G$, $X \circlearrowleft G$ is trivial
s.t. π is right G-inv.
b) π is locally trivial wrt the topology.

$\forall p \in X$, you can find Zariski open nbd $U \ni x$
and an étale cover $U' \xrightarrow{e} U$ s.t.

$e^* E|_U \cong U' \times_U E|_U$ trivial G-bundle.

Rmk: G is typically smooth linear alg group so we have no need to use a stronger site like fpqc.
In the case $G = GL_n$ & X is a smooth curve, Zariski site is enough.

For families, it's important to use étale site.

Expl: totalspace of locally freesheaf $\longrightarrow X_{Zar}$

$$\begin{array}{ccc} \text{totalspace of locally freesheaf} & \longrightarrow & X_{Zar} \\ \uparrow & & \cup \\ U \times GL_n & \xrightarrow{pr_1} & U \ni x \end{array}$$

orthogonal group

Expl: O_r-bundle E = GL_r-bundle w/ $\sigma : E \xrightarrow{\sim} E^*$.
E on X_S can have each $E|_{X_p}$ be an O_r-bundle but
E fails to be an O_r-bundle.

Bundles associated to Princ. G-Bundles (G-torsors).

Expl: $F = k^n$ dim n vectors space over k.
$G := GL(F) = GL_n(k)$.
I know $G \curvearrowright F$ in the obvious.

Suppose I give you a G-bundle E.

$$E(F) := E \times_G F = E \times F / G$$

where $g \cdot (e,f) = (eg^{-1}, gf)$.

$E(F)$ is now a rank n vector bundle.

Conversely if V is rank n vector bundle, we get a G-bundle by

$$\underline{\mathcal{I}som}_{\mathcal{O}_X}(\mathcal{O}_X^n, V) \quad \text{"associated frame bundle"}$$

Def: G-bundle E

$$\left\{ \begin{array}{l} F \text{ quasiproj scheme}/k, \\ G \curvearrowright F \end{array} \right\} \overset{E(-)}{\rightsquigarrow} \left\{ \begin{array}{l} E(F) = E \times_G F \text{ (scheme)} \\ g(e,f) = (eg^{-1}, gf) \end{array} \right\}$$

$E(F)$ "associated bundle w/ fibre F"

Extension of Structure Groups:

Let $\rho: G \to H$ morphism of alg groups. ("linear alg groups G $G \hookrightarrow GL_n$")

Given a G-bundle E, I can form an H-bundle

$$E(H) = E \times_G H \quad \left(\text{quotient by } g(e,f) = (e \cdot g^{-1}, \rho(g)h)\right).$$

$$\rightsquigarrow \quad H^1_{\text{ét}}(X, G) \longrightarrow H^1_{\text{ét}}(X, H)$$

($H^1_{\text{ét}}(X,G)$ classifies principal G-bundle)

(Recall: $H^1(X, \mathcal{O}_X^*) = \mathrm{Pic}(X)$
X sep.)

Reduction of Structure Groups

If F is an H-bundle, $G \hookrightarrow H$ is closed immersion + G is alg grp, then does F come from G? i.e.

$\exists$ a G-bundle E and an iso

$$\tau: E(H) \xrightarrow{\sim} F$$
$$E(H) = E \times_G H$$

Expl: Given $\mathcal{O}_X^{\oplus n}$ and view as a GL_n-bundle.
You can see that it comes from a $\mathbb{G}_m^{\oplus n}$-bundle.

Lemma: (Exercise) If $\rho: G \hookrightarrow H$ closed immersion and F is an H-bundle, then consider $H \twoheadrightarrow H/G$ and $F(H/G) := F/G$.

$\exists$ a natural bijection

$$\left\{ \begin{array}{c} \text{sections} \\ \sigma: X \longrightarrow F/G \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{reduction of str.} \\ \text{group of } F \\ \text{from } H \text{ to } G \end{array} \right\}.$$

Review of Algebraic Stacks

Def: An alg. stack is a stack $\mathfrak{X}/S_{fppf}$ (S scheme) s.t.

1) $\Delta_{\mathfrak{X}}$ is representable (by alg spaces).

2) $\exists$ a smooth atlas i.e. a scheme X w/ smooth surj. map $\rho: X \longrightarrow \mathfrak{X}$.

$\mathfrak{X}$ is smooth if it has a smooth atlas $\rho: X \to \mathfrak{X}$ s.t. X is smooth.

$$\dim \mathfrak{X} := \dim X - \dim X \times_{\rho, \mathfrak{X}, \rho} X$$

$$\begin{tikzcd} X \times_{\rho, \mathfrak{X}, \rho} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, "\rho"] \\ X \arrow[r, "\rho"'] & \mathfrak{X} \end{tikzcd}$$

Expl: $BGL_n = [*/GL_n]$

$$\dim BGL_n = -\dim GL_n = -n^2.$$

Goal (short term).

Theorem: X a smooth alg curve, G reductive group
$\mathrm{Bun}_G(X)$ — stack of princ. G-bundles

Then

1) $\mathrm{Bun}_G(X)$ is an algebraic stack
2) $\mathrm{Bun}_G(X)$ is smooth
3) $\dim \mathrm{Bun}_G(X) = (g(X)-1)\cdot \dim G$.

Expl: $\mathrm{Bun}^0_{GL_1} C \cong \mathrm{Pic}^0_{GL_1} C \times B\mathbb{G}_m$ ((C has genus 0))

$\cong \{*\} \times B\mathbb{G}_m \leftarrow$ has dim $0+(-1)$
$= -1.$

Formula $(0-1)\cdot 1 = -1.$ ✓

Long Term Goals (?):

1) Compute line bundles on $\mathrm{Bun}_G(X)$ —
(when G is simple).

2) What is the cohomology of $\mathrm{Bun}_G(X)$ —
$(H^*(\mathrm{Bun}_G(X), \overline{\mathbb{Q}}_\ell))$.

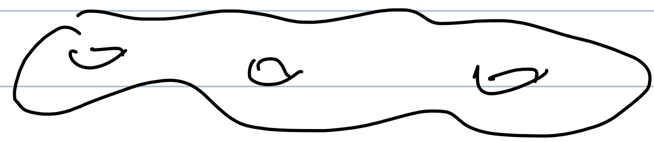
Wednesdays.

In 2 weeks:

- discuss topological side..
- Work out examples (e.g. $\mathrm{Bun}_{GL_2}(\mathbb{P}^1)$)
- $\mathrm{Bun}^{ss}_{GL_2}(\mathbb{P}^1)$
- $\mathrm{Bun}^{ss}_{GL_2}(E)$ E elliptic curve..

What's the
Topological Classification of G-bundles?

$$\mathcal{M}^{top}_G(X) \cong \pi_1(G) \overset{G=GL_n}{=} \mathbb{Z}$$

X 

$c_1(E)$ in alg. geo.