# Topological Classification of $G$-bundles on a Riemann surface and Examples of $Bun_G$

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# Topological Classification of $G$-Bundles

For this section, all maps are continuous and taken up to homotopy. Let $X$ be a compact connected orientable surface and $G$ a connected topological group.

**Theorem**

Let $x_0 \in X$ be a point, $D$ a small disc containing $x_0$, $D^* := D \setminus \{x_0\}$, $X^* := X \setminus \{x_0\}$. Let $\mathcal{M}_G(X)$ be the set of $G$-bundles on $X$. Denote $L_G := \{g : D^* \to G\}$, $Triv_{X^*} := \{f : X^* \to G\}$, $Triv_D := \{h : D \to G\}$. Then

$$\mathcal{M}_G(X) \simeq Triv_{X^*} \backslash L_G / Triv_D.$$

**Corollary**

$\mathcal{M}_G(X) \simeq \pi_1(G)$.

# A Topological Lemma

The proof relies on the following lemma, which can be seen as an analog of the Poincaré lemma.

**Lemma (Homotopy Invariance)**

Let $Y$ be a sufficiently nice topological space (e.g. locally compact Hausdorff second countable). Let $E \to Y \times [0,1]$ be a principal $G$-bundle. Then the restrictions $E_0 \to Y \times \{0\}$ and $E_1 \to Y \times \{1\}$ are isomorphic.

For an idea of the proof, one first claims there is an open cover $\{U_\alpha\}$ of $Y$ and a finite collection of open intervals $\{I_k\}$ such that $E$ is trivial over $U_\alpha \times I_k$ where $\{I_k\}$ cover $[0,1]$. Then one uses a partition of unity argument to glue the transition functions together to get the required identification.

# Proof Sketch of Theorem

- Recall that a $G$-bundle $F$ on $X$ is determined by gluing over an open cover. Take the cover $\{X^*, D\}$.
- Claim that a $G$-bundle on $D$ or $X^*$ is trivial. The lemma can show this for $D$, and reduces the case of $X^*$ to a wedge of circles; in this case contractibility of the universal cover and connectedness of $G$ imply the result.
- Thus, the class of the bundle $F$ is determined by a cocycle over $D^*$. $L_G(X)$ precisely parametrizes this set.
- But there is a redundancy given by changing trivializations over $D, X^*$. This gives the double quotient as in the theorem.

**Remark**

This double coset description of $\mathcal{M}_G(X)$ will appear again later in the algebraic context, in Weil's uniformization theorem.

# Proof Sketch of Corollary

- Let $\delta$ be the generator of $\pi_1(D^*)$ and consider the map

$$[g : D^* \to G] \mapsto g_*(\delta) \in \pi_1(G).$$

- To prove: it is well-defined on $Triv_{X^*} \backslash L_G / Triv_D$ (i.e. that changing trivializations defines the same map) and that it is injective on $Triv_{X^*} \backslash L_G / Triv_D$.
- $h : D \to G$ restricted to $D^*$ is homotopically trivial. For $f : X^* \to G$, observe that $D^* \hookrightarrow X^*$ lies in $[\pi_1(X^*), \pi_1(X^*)]$ (it gets trivialized by the 2-cell of $X$); hence acting by $Triv_{X^*}$ changes $g$ by an element of $[\pi_1(G), \pi_1(G)] = id$ because $\pi_1(G)$ is abelian.
- For injectivity, observe that $g_*(\delta) = id$ implies $g$ extends to $D$.

# Relevance to $Bun_G(X)$

For $S$ any space, the topological classification gives $\pi_0(Bun_G(S))$. For intuition, view $Bun_G(S) = Maps(S \to BG)$. Two maps are connected exactly when they are homotopic, giving the interpretation. Thus, returning to $X$ a compact Riemann surface, specifying a connected component of $Bun_G(X)$ amounts to fixing an element of $\pi_1(G)$.

**Example: $G = GL_n(\mathbb{C}), SL_n(\mathbb{C}), PGL_n(\mathbb{C})$.**

Identifying $GL_n$-bundles with vector bundles of rank $n$, we obtain that connected components of $Bun_n(X)$ are identified with $\mathbb{Z}$, corresponding to a choice of degree.
$SL_n$ is simply connected so the moduli space is connected.
$PGL_n$ has an $\mathbb{Z}/n\mathbb{Z}$-cover by $SL_n$, so "degree" is $\mathbb{Z}/n\mathbb{Z}$-valued.

# Example: $Bun_{GL_1}(X)$

Let $X$ be a smooth projective curve.

**Theorem**

$$Bun_{GL_1}(X) \cong \coprod_{d\in\mathbb{Z}} (Pic^d(X) \times B\mathbb{G}_m).$$

**Beginning of proof: Definitions.**

Using line bundles, recall:

$$Bun_{GL_1}(X)(S) = \{S\text{-flat line bundles on } X \times S\}$$

$$B\mathbb{G}_m(S) = \{\text{line bundles on } S\}$$

$$Pic^d(X)(S) = \{S\text{-flat line bundles of degree } d \text{ on } X \times S\}/Pic(S).$$

## $Bun_{GL_1}(X)$: More Definitions

Fix $S$ a $\mathbb{C}$-scheme and suppose $T$ is a covering of $S$.

- Previous slide defined objects. Observe that morphisms behave as follows:
- For $u, v \in B\mathbb{G}_m(S)$, $Isom(u,v)(T) = \Gamma(T, \mathcal{O}_T)^\times \simeq \mathbb{G}_m(T)$ (an isomorphism of line bundles is a section of $\mathbb{G}_m$).
- For $u, v \in Bun_{GL_1}(X)(S)$, $Isom(u,v)(T)$ is isomorphisms of line bundles again. Thus $Isom(u,v)(T) \simeq \mathbb{G}_m(T)$.
- $Pic^d(X)$ only has identity morphisms.

# $Bun_{GL_1}(X)$: Conclusion of Proof

Define a map

$$\coprod_{d \in \mathbb{Z}} (Pic^d(X)(S) \times B\mathbb{G}_m(S)) \to Bun_{GL_1}(X)(S)$$

as follows. First, choose a representative $\mathcal{L} \in Bun_{GL_1}(X)(S)$ of each isomorphism class $L \in Pic^d(X)(S)$, for all $d$. Then on objects, define:

$$(L, M) \mapsto \mathcal{L} \otimes M$$

and on morphisms:

$$(id, \tau) \mapsto \tau.$$

It is clearly an isomorphism on morphisms. On objects, there is a bijection between the $L$ and the $\mathcal{L}$, and $\{\mathcal{L} \otimes M, M \in Pic(S)\}$ is the equivalence class of $\mathcal{L}$. $\square$

# Some Further Remarks on $Bun_{GL_1}(X)$

- The map above defines an action of the objects of $B\mathbb{G}_m$ on $Bun_{GL_1}(X)$ over $\coprod Pic^d(X)$. This makes $Bun_{GL_1}(X)$ a (trivial) *gerbe* over $\coprod Pic^d(X)$, *banded by* $\mathbb{G}_m$. More generally, for $\mathcal{A}$ is a sheaf of abelian groups, a *gerbe banded by* $\mathcal{A}$ is a map $X \to Y$ of stacks locally looks like the *trivial gerbe* $U \times B\mathcal{A} \to U$.
- $Bun_{GL_1}(X)$ is almost a scheme – we don't need stacks to make sense of it! This is because the automorphism groups of line bundles are well controlled.
- In general, imposing a *stability condition* can isolate substacks of $Bun_G(X)$ that are similarly close to schemes (precisely, they are gerbes over $B\mathbb{G}_m$).

# Vector Bundles on $(\mathbb{P}^1)$

With the aid of the following theorem, we can make the structure of $GL_n$-bundles on $\mathbb{P}^1$ reasonably explicit:

**Theorem (Grothendieck)**

For any vector bundle $F$ on $\mathbb{P}^1$, there is a unique (though non-canonical) splitting

$$F \cong \bigoplus_{i=1}^{n} \mathcal{O}(\lambda_i)^{m_i}$$

where $\lambda_i$ are decreasing integers.

There is an elementary proof via linear algebra and a cohomological one.

# Example: $Bun_{GL_2}(\mathbb{P}^1)$

There is an action of $Bun_{GL_1}(\mathbb{P}^1) \cong \coprod_{d\in\mathbb{Z}} B\mathbb{G}_m$ on $Bun_{GL_2}(\mathbb{P}^1)$ by tensoring with a family of line bundles. This gives an isomorphism of the degree-$n$ component with the degree-$n+2d$ component, so it is sufficient to understand degrees $0, 1$. We consider degree 0 for convenience: Denote this space $Bun_{2,0}(\mathbb{P}^1)$.
Using Grothendieck's theorem, the isomorphism classes of $\mathbb{C}$-points of $Bun_{2,0}(\mathbb{P}^1)$ are in bijection with $\mathbb{Z}_{\geq 0}$ via

$$m \in \mathbb{Z}_{\geq 0} \mapsto \mathcal{O}(m) \oplus \mathcal{O}(-m).$$

**Remark.**

This is actually the same as $Bun_{SL_2}(\mathbb{P}^1)$!

# $Bun_{2,0}(\mathbb{P}^1)$: Automorphisms of $\mathbb{C}$-points

First, consider

$$End(\mathcal{O}(m) \oplus \mathcal{O}(-m)) \simeq \Gamma(\mathbb{P}^1, \mathcal{O}(2m) \oplus \mathcal{O}^{\oplus 2} \oplus \mathcal{O}(-2m)).$$

We want the everywhere-invertible sections. For $m = 0$, this is clearly a copy of $GL_2$.
For $m > 0$, observe that $\mathcal{O}(-2m)$ has no global sections and that any global section of $\mathcal{O}(2m)$ has zeros. Writing our endomorphisms as matrices, we see that they take the form

$$\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \text{ with } a, d \in \Gamma(\mathbb{P}^1, \mathcal{O}), \; b \in \Gamma(\mathbb{P}^1, \mathcal{O}(2m))$$

where $b$ can be 0. Thus $a$ and $d$ cannot be 0 but $b$ can be any section, so the automorphism group is $(\mathbb{C}^*)^{\times 2} \ltimes \mathbb{C}^{2m+1}$.

## $Bun_{2,0}(\mathbb{P}^1)$: Extensions and Stackiness

The existence of non-trivial extensions of line bundles reveals the fundamentally "stacky" nature of $Bun_{2,0}(\mathbb{P}^1)$. Recall $Ext^1(\mathcal{F}, \mathcal{G})$ is the vector space of equivalence classes of extensions

$$0 \to \mathcal{G} \to \mathcal{E} \to \mathcal{F} \to 0,$$

with 0 element representing $\mathcal{F} \oplus \mathcal{G}$.

$$Ext^1(\mathcal{O}(m), \mathcal{O}(-m)) \simeq H^1(\mathcal{O}(-2m)) \simeq H^0(\mathcal{O}(2m-2)),$$

which has dimension $2m-1$ for $m \neq 0$. It can be shown that this implies existence of families on $\mathbb{A}^1 \times \mathbb{P}^1$ where the fiber over 0 is $\mathcal{O}(m) \oplus \mathcal{O}(-m)$ but all other fibers are isomorphic to a different isomorphism class.

# $Bun_{2,0}(\mathbb{P}^1)$: "Stacky" Families of Extensions

**Example.**

A rank-2 vector bundle on $\mathbb{P}^1$ is equivalent to a $2 \times 2$ transition matrix $\sigma$ over $\mathbb{C}[t, t^{-1}]$. This can be put canonically in upper triangular form, with splitting coming from reducing the matrix to diagonal. Consider the family corresponding to the matrices

$$\begin{bmatrix} t^{-m} & \lambda \\ 0 & t^m \end{bmatrix}$$

If $\lambda = 0$, it splits as $\mathcal{O}(-m) \oplus \mathcal{O}(m)$; if $\lambda \neq 0$, we get $\mathcal{O} \oplus \mathcal{O}$.

**Remark.**

This shows that $\mathcal{O} \oplus \mathcal{O}$ degenerates into all isomorphism classes.

# Further Remarks

- We have ignored automorphisms of the base curve in thinking about these spaces. This does not become an issue if we fix a base curve, but if we want to think about these moduli spaces in families of curves, it would become relevant. This would amount to thinking about vector bundles on $\mathcal{M}_g$, the moduli stack of genus $g$ curves!
- $\mathbb{P}^1$ is very special – the structure of vector bundles is both reasonably explicit and gives rise to some "discreteness" on the level of $\mathbb{C}$-points. Already for elliptic curves, the structure is much more involved.
- However, the fact that these stacks represent functors whose section categories have extra structure allow us to say a lot about the structure of sheaves and cohomology on them, with connections to number theory, PDE, and physics.

# Looking Ahead

- Next time, we will see a presentation of $Bun_G(X)$ as an algebraic stack, giving us more tools to study its geometry.
- This will be deeply connected to the differential-geometric/physics and arithmetic perspective on these spaces, which we will hopefully discuss – see Atiyah-Bott's 1982 paper "The Yang-Mills Equations on a Riemann Surface" and Harder-Narasimhan's 1975 paper "On the cohomology groups of Moduli Spaces of Vector Bundles on Curves".

Thank you!