Categorical Logic for Realizability, part I: Categories & the Yoneda Lemma

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Introduction

The Greater Plan

This talk is the first in a series meant to work our way up to understanding the basic ideas behind the realizability toposes of Pitts, Hyland, and Johnstone. We will break up the material into three parts.

1. Category theory, particularly the notions of representability and adjunction (The current slides.)
2. Fibrations and toposes
3. PCAs and realizability toposes

The Lesser Plan

This will be a brisk introduction to categories, and will be light on logic. We will try to cover

- Categories
- Functors & natural transformations
- Representability & the Yoneda Lemma
- Adjunctions
- Limits & colimits

Why Categories?

Category theory has roots in work in algebraic topology by Samuel Eilenberg and Saunders Mac Lane (see [Eilenberg and MacLane, 1945]) where the focus was on the auxiliary gear needed to describe natural transformations.

Some patterns come up repeatedly in different settings. Categories as a lingua franca.

- Free groups, discrete topological spaces, and vector spaces are all share a common universal property relative to **Set**.
- The basic semantics for first order logic, Martin-Löf type theory, and Boolean-valued models of set theory all live in different forms of the same sort of structure.

Category theory offers tools for transporting reasoning to different settings by offering a variety of setting-independent notions for reasoning about disparate kinds of objects.

Categories

Categories

A category $\mathcal{C}$ consists of the following data:

- A class C_0 whose elements are called *objects*,
- For any $X, Y \in C_0$ a set $\mathcal{C}(X, Y)$ of *morphisms* $X \to Y$,
- For any $X, Y, Z \in C_0$ a function $c_{XYZ} : \mathcal{C}(Y, Z) \times \mathcal{C}(X, Y) \to \mathcal{C}(X, Z)$ called *composition* (usually writing $c_{XYZ}(f, g)$ as $f \circ g$).
- For each $X \in C_0$ an element $1_X \in \mathcal{C}(X, X)$,

satisfying associativity of composition, and that 1_X is a left and right identity element under composition.

In practice we treat the $\mathcal{C}(X, Y)$ as disjoint; that is we may think of morphisms as really being elements of $\sum_{X, Y \in C_0} \mathcal{C}(X, Y)$.

Categories

Examples of categories:

- Categories of common mathematical objects and their appropriate notion of "mapping"
  - **Set**: sets and functions
  - **FinSet**: finite sets and functions
  - **Ring**: rings and ring homomorphisms
  - $R - \mathbf{Mod}$: R-modules and module homomorphisms
  - **Top**: topological spaces and continuous functions
  - **HTop**: topological spaces and homotopy-equivalence classes of maps
  - **Pos**: posets and order-preserving maps

Categories

More abstract categories drawn from common structures

- A set X can be thought of as a category whose only morphisms are identities
- A group G can be thought of as a category with one object with all morphisms invertible
- A preorder can be thought of as a category $\mathcal{C}$ in which $\mathcal{C}(X, Y)$ has at most one element for all $X, Y \in C_0$
- $\mathcal{C}^{op}$: the opposite category of $\mathcal{C}$.
    - $C_0^{op} = C_0$
    - $\mathcal{C}^{op}(X, Y) = \mathcal{C}(Y, X)$
    - $c_{XYZ}^{\mathcal{C}^{op}} = c_{ZYX}^{\mathcal{C}} \circ \tau$ with τ the twist map $\mathcal{C}(Z, Y) \times \mathcal{C}(Y, X) \to \mathcal{C}(Y, X) \times \mathcal{C}(Z, Y)$.

In any category we can define the notion of isomorphism, and it's what you would expect. A morphism $f : X \to Y$ is an isomorphism if there's a $g : Y \to X$ with $fg = 1_Y$ and $gf = 1_X$.

The most basic generalizations of injective and surjective maps are monomorphisms and epimorphisms.

- A morphism f is a **monomorphism** if $fg = fh$ implies $g = h$ for all g, h.
- We say f is an **epimorphism** if $gf = hf$ implies $g = h$ for all g, h.

Monomorphism + epimorphism $\not\Rightarrow$ isomorphism (counterexample: $(\mathbb{N}, +)$). The converse does hold.

Functors & Natural Transformations

Functors & Natural Transformations

Functors are the natural notion of a map between categories. A **functor** $F : \mathcal{C} \to \mathcal{D}$ consists of

- A function $F_0 : C_0 \to D_0$
- For every $X, Y \in C_0$ a function $F_{X,Y} : \mathcal{C}(X, Y) \to \mathcal{D}(F_0X, F_0Y)$

such that identities and composition are preserved.

Note that this means all functors will preserve isomorphisms.

It is common practice to simply write FX or $Ff : FX \to FY$ rather than write out all of the subscripts.

Functors & Natural Transformations

Example functors:

- $id : \mathcal{C} \to \mathcal{C}$ taking every object and morphism to itself.
- π_1 : **pTop** $\to$ **Grp** the fundamental group functor (from pointed topological spaces).
- $(-)^{\mathrm{ab}}$: **Grp** $\to$ **Grp**, the abelianization functor.
- $\mathcal{P}(-)$: **Set**^{op} $\to$ **Set**, taking each set to its power set, and each function f to $f^{-1}[-]$.
- Given a functor $F : \mathcal{C} \to \mathcal{D}$, there is its opposite functor $F^{op} : \mathcal{C}^{op} \to \mathcal{D}^{op}$ defined by exactly the same data as F.

Functors worth singling out:

- $\mathcal{C}(X, -) : \mathcal{C} \to \mathbf{Set}$.
  - On objects, $Y \mapsto \mathcal{C}(X, Y)$.
  - On morphisms, $f : Y_1 \to Y_2$ in $\mathcal{C}$ goes to the function $\mathcal{C}(X, f)$ with $g \in \mathcal{C}(X, Y_1) \mapsto fg \in \mathcal{C}(X, Y_2)$.

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- $\mathcal{C}(-, X) : \mathcal{C}^{op} \to \mathbf{Set}$.
    - On objects, $Y \mapsto \mathcal{C}(Y, X)$.
    - On morphisms, $f : Y_1 \to Y_2$ goes to the function $\mathcal{C}(f, X)$ given by $g \in \mathcal{C}(Y_2, X) \mapsto gf \in \mathcal{C}(Y_1, X)$.

These play a special role we'll see later.

Functors & Natural Transformations

Let F, G be functors $\mathcal{C} \to \mathcal{D}$. A **natural transformation** $\alpha : F \to G$ is an assignment

$$X \in C_0 \mapsto \alpha_X \in \mathcal{D}(FX, GX)$$

such that for any $X, Y \in \mathcal{C}$ and $f : X \to Y$, the square

$$\begin{array}{ccc} FX & \xrightarrow{\alpha_X} & GX \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ FY & \xrightarrow[\alpha_Y]{} & GY \end{array}$$

commutes.

We call the morphism α_X the *component of* α *at* X.

Functors & Natural Transformations

Examples of natural transformations:

- For any $F : \mathcal{C} \to \mathcal{D}$, $X \mapsto 1_{FX}$ is a natural transformation $F \to F$.
- $c : id_{\mathbf{Grp}} \to (-)^{\mathrm{ab}}$, whose components are the canonical projections $G \to G^{\mathrm{ab}}$.
- A group action of G may be viewed as a functor $G \to \mathbf{Set}$. If $A, B : G \to \mathbf{Set}$ are two group actions, then a natural transformation $\alpha : A \to B$ is exactly a G-equivariant function.
- Let $\mathcal{P}(-) : \mathbf{Set} \to \mathbf{Set}$ be the covariant power set functor. Then the singleton functions $\iota_X : X \to \mathcal{P}(X)$ constitute a natural transformation $1_{\mathbf{Set}} \to \mathcal{P}(-)$.
- The maps sending an element g of a group G to the unique $\phi : \mathbb{Z} \to G$ with $\phi(1) = g$ are the components of a natural transformation $U \to \mathbf{Grp}(\mathbb{Z}, -)$, where $U : \mathbf{Grp} \to \mathbf{Set}$ is the forgetful functor.

Given functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$, the data

- $(G \circ F)_0 := G_0 \circ F_0$
- $(G \circ F)_{X,Y} := G_{X,Y} \circ F_{X,Y}$

constitute a functor. This operation is associative because function composition is.

With the identity functors, small categories and their functors form a category.

Functors & Natural Transformations

$$\begin{array}{ccccc} FX & \xrightarrow{\alpha_X} & GX & \xrightarrow{\beta_X} & HX \\ {\scriptstyle Ff}\downarrow & & {\scriptstyle Gf}\downarrow & & \downarrow{\scriptstyle Hf} \\ FY & \xrightarrow[\alpha_Y]{} & GY & \xrightarrow[\beta_Y]{} & HY \end{array}$$

Given functors $F, G, H : \mathcal{C} \to \mathcal{D}$ and natural transformations $F \stackrel{\alpha}{\to} G \stackrel{\beta}{\to} H$, the assignment $X \in \mathcal{C} \mapsto \beta_X \circ \alpha_X$ is also a natural transformation $F \to H$. This is because pastings of commutative squares are commutative.

So for categories $\mathcal{C}, \mathcal{D}$, there is a category whose objects are functors, and whose morphisms are natural transformations. This is the **functor category** $\mathcal{D}^{\mathcal{C}}$.

An isomorphism in such a category is a **natural isomorphism**.

Functor Categories & The Yoneda Lemma

Functor categories are ubiquitous.

- $\mathbf{Set}^{\Delta^{op}}$, the category of simplicial sets (important in abstract homotopy theory)
- $\mathbf{Set}^{G}$, the category of group actions of a fixed group, with equivariant maps as morphisms.
- $\mathbf{Vec}_{\mathbb{C}}^{G}$ for a fixed group G is the category of complex representations of G.
- The category of groups, and a number of other "algebraic" categories, can be viewed as the full subcategory of Set^{T} of finite limit preserving functors from a *Lawvere theory* T.
- Chain complexes and their morphisms constitute a subcategory of $R - \mathrm{Mod}^{\mathbb{Z}}$, where we view $\mathbb{Z}$ as a partial order.

The Yoneda Lemma

Functor Categories & The Yoneda Lemma

Functors of the form $\mathcal{C}(-, X)$ have a special place in category theory. This is because of the **Yoneda lemma**, which states that there is a natural isomorphism

$$\mathbf{Set}^{\mathcal{C}^{op}}(\mathcal{C}(-, X), F) \simeq FX$$

for any $F : \mathcal{C}^{op} \to \mathbf{Set}$. Here we consider these as functors $\mathcal{C}^{op} \times \mathbf{Set}^{\mathcal{C}^{op}} \to \mathbf{Set}$.

A consequence of this is the **Yoneda embedding**. Letting $F = \mathcal{C}(-, Y)$, we have

$$\mathbf{Set}^{\mathcal{C}^{op}}(\mathcal{C}(-, X), \mathcal{C}(-, Y)) \simeq \mathcal{C}(X, Y)$$

giving an embedding functor $\mathcal{C} \to \mathbf{Set}^{\mathcal{C}^{op}}$.

Proof (sketch) of the Yoneda Lemma

Certainly a natural transformation $\alpha : \mathcal{C}(-, X) \to F$ picks out some element of FX—namely, $\alpha_X(1_X) \in FX$.

Given $a \in FX$, let $\xi^a_Y : \mathcal{C}(Y, X) \to FY$ be given by the action

$$f \in \mathcal{C}(Y, X) \mapsto Ff(a) \in FY.$$

In fact this is the *only* possible natural transformation sending 1_X to a, otherwise

$$\begin{array}{ccccccc} 1_X & & \mathcal{C}(X,X) & \xrightarrow{\mathcal{C}(f,X)} & \mathcal{C}(Y,X) & & f \\ \big\downarrow & & {\scriptstyle \xi^a_X}\big\downarrow & & \big\downarrow{\scriptstyle \xi^a_Y} & & \big\downarrow \\ a & & FX & \xrightarrow[Ff]{} & FY & & ? \end{array}$$

doesn't even commute.

Proof (sketch) of the Yoneda Lemma

So for a given X and F,

- $A_{X,F} : \mathbf{Set}^{\mathcal{C}^{op}}(\mathcal{C}(-,X),F) \to FX$ given by $\alpha \mapsto \alpha_X(1_X)$
- $B_{X,F} : FX \to \mathbf{Set}^{\mathcal{C}^{op}}(\mathcal{C}(-,X),F)$ given by $a \mapsto \xi^a$.

They are natural in both X and F. The proofs are not difficult. E.g. for the naturality of A in the variable X, given $f : Y \to X$

$$\begin{aligned} Ff \circ A_{X,F}(\alpha) &= Ff(\alpha_X(1_X)) \\ &= \alpha_Y(f) \\ &= \alpha_Y \circ \mathcal{C}(Y,f)(1_Y) \\ &= A_{Y,F}(\alpha \circ \mathcal{C}(-,f)) \\ &= A_{Y,F} \circ \mathbf{Set}^{\mathcal{C}^{op}}(\mathcal{C}(-,f),F)(\alpha) \end{aligned}$$

Proof (sketch) of the Yoneda Lemma

The above series of identities serves to establish that the below square commutes.

$$\begin{array}{ccc}
\mathbf{Set}^{\mathcal{C}^{op}}(\mathcal{C}(-,X),F) & \xrightarrow{A_{X,F}} & FX \\
{\scriptstyle \mathbf{Set}^{\mathcal{C}^{op}}(\mathcal{C}(-,f),F)}\big\downarrow & & \big\downarrow{\scriptstyle Ff} \\
\mathbf{Set}^{\mathcal{C}^{op}}(\mathcal{C}(-,Y),F) & \xrightarrow[A_{Y,F}]{} & FY
\end{array}$$

Illustrating Yoneda

Some examples:

- Cayley's theorem in group theory is a consequence. There's an isomorphism $\mathbf{Set}^G(G(*,-),G(*,-)) \simeq G$ by Yoneda.
- The edges of a directed multigraph X correspond to morphisms from $\bullet \to \bullet$, and the vertices to morphisms from $\bullet$. Directed multigraphs are **Set**-valued functors on $\Gamma = \{E \rightrightarrows V\}$; these special graphs are $\Gamma(E,-)$ and $\Gamma(V,-)$.
- The simplicial sets $\Delta(n) := \Delta(-,n)$ are generic n-simplices. Given a simplicial set S, the elements of $S(n)$ correspond to mappings from $\Delta(n)$.

Representability

When a functor $F \in \mathbf{Set}^{\mathcal{C}^{op}}$ (resp. $\mathbf{Set}^{\mathcal{C}}$) is isomorphic to one of the form $\mathcal{C}(-, X)$ (resp. $\mathcal{C}(X, -)$), we say it is **representable**, and call X the *representing object* of F.

- Let G be a group and H a normal subgroup. Let $F = G_H : \mathbf{Grp} \to \mathbf{Set}$ be the functor taking I to the set of group homomorphisms $G \to I$ that kill H. This is representable if there's a group Q with

$$\mathbf{Grp}(Q, -) \simeq G_H.$$

  That is *arbitrary* homomorphisms $Q \to I$ correspond to H-killing homomorphisms $G \to I$. There always is such a representing object: G/H.

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- Let X be an arbitrary set and let $F = \mathbf{Set}(X, |-|) : \mathbf{Top} \to \mathbf{Set}$ be the functor taking a space T to the set of arbitrary functions $X \to |T|$ (where $|T|$ is the underlying set of T).

  Representability means a topological space S such that continuous functions $S \to T$ correspond to arbitrary functions $X \to |T|$. Where S is the discrete space on X (D_X for the nonce), we do have

  $$\mathbf{Top}(D_X, -) \simeq \mathbf{Set}(X, |-|).$$

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- Fix a set A and let $F = \mathcal{P}(A \times -) : \mathbf{Set}^{op} \to \mathbf{Set}$ take a set X to the set $\{R \mid R \subset A \times X\}$, and take morphisms $f : Y \to X$ to the functions $R \mapsto (1_A \times f)^{-1}[R]$ (inverse image long $(a, y) \mapsto (a, f(y))$). This is representable by

$$\mathbf{Set}(-, \mathcal{P}(A)) \simeq \mathcal{P}(A \times -)$$

- Let $F = \mathcal{O} : \mathbf{Top}^{op} \to \mathbf{Set}$ take a space X to its set of open sets, and a continuous function to the inverse image operation on open sets. This functor is representable by the Sierpinski space Σ

$$\mathbf{Top}(-, \Sigma) \simeq \mathcal{O}$$

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- Let $(X_i)_{i \in I}$ be an indexed family of objects of $\mathcal{C}$, and let $\mathrm{Cone}_{X_i} : \mathcal{C}^{op} \to \mathbf{Set}$ be the functor with $\mathrm{Cone}_{X_i}(Y)$ the set of all $(f_i : Y \to X_i)_{i \in I}$, and morphisms acting by precomposition.

  Then a representing object for Cone_{X_i} is exactly a *product* of the X_i. This is the cartesian product in **Set**, direct sums in **Grp**, **Ab**, $\mathbf{Vec}_k$, infima in partial orders, etc. Generally denoted $\prod_{i \in I} X_i$.

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- Let M, N be modules for a commutative ring R. Let $\mathrm{Bil}_{M,N} : R - \mathbf{Mod} \to \mathbf{Set}$ take P to the set of bilinear maps $M \times N \to P$, with action on module homomorphisms $P \to Q$ given by composition.
  This functor is represented by $M \otimes N$ with

$$R - \mathbf{Mod}(M \otimes N, -) \simeq \mathrm{Bil}_{M,N}$$

Representability

Recall the importance of 1_X in determining the structure of natural transformations from $\mathcal{C}(-, X)$. An isomorphism $\mathcal{C}(-, X) \simeq F$ determines a universal/generic element of FX.

- $\mathbf{Grp}(G/H, -) \simeq G_H$ above determines a universal H-killing homomorphism in $G_H(G/H)$, the quotient projection.

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- $\mathbf{Top}(-, \Sigma) \simeq \mathcal{O}$ determines a generic open set in $\mathcal{O}(\Sigma)$: $\{1\}$.
- $R - \mathbf{Mod}(M \otimes N, -) \simeq \mathrm{Bil}_{M,N}$ determines a generic bilinear map $M \times N \to M \otimes N$.

Representability

Some facts about representability:

- Representing objects are unique up to isomorphism. This is a consequence of the Yoneda embedding.

$$\mathcal{C}(-, A) \simeq F \simeq \mathcal{C}(-, B)$$

- A functor $F : \mathcal{C}^{op} \to \mathbf{Set}$ is representable if and only if there is an $X \in \mathcal{C}$ and $x \in F(X)$ such that for every $Y \in \mathcal{C}$ and $y \in F(Y)$ there is a unique $f : Y \to X$ with $Ff(x) = y$. (Universal element induced by identity)

Adjoint Functors

Adjoint Functors

We dive right into the definition(s). Given functors

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$$

we say that F **is left adjoint to** G, G **is right adjoint to** F, or $F \dashv G$, if and only if for every $X \in \mathcal{C}$ the functor $\mathcal{C}(F-, X)$ is representable by GX.

Alternatively if $\mathcal{C}(FY, X) \simeq \mathcal{D}(Y, GX)$, natural in X and Y (i.e. isomorphic as functors $\mathcal{D}^{op} \times \mathcal{C} \to \mathbf{Set}$).

Adjoint Functors

Some examples:

- $(-)^{ab} : \mathbf{Grp} \leftrightarrows \mathbf{Ab} : I$, where I is the inclusion of **Ab** into **Grp**.
- $F : \mathbf{Set} \leftrightarrows \mathbf{Grp} : U$, where U "forgets" the group structure and F takes a set to the free group on that set.
- If P is a poset and $P \times P$ the product poset, then the function $p \mapsto \langle p, p \rangle$, considered as a functor, has a right adjoint if and only if P has binary meets; it has a left adjoint iff it has binary joins.
- Any functor $\mathbf{1} \to \mathbf{Set}$ whose image is a singleton set has a right adjoint.
- In **Set** the functor $X \times -$ has right adjoint $(-)^X$ for all X.

Adjoint Functors

The natural isomorphism

$$\mathcal{C}(FY, X) \simeq \mathcal{D}(Y, GX)$$

gives us a special morphism $\eta_Y : Y \to GFY$ for all $Y \in \mathcal{D}$, corresponding to the map 1_{FY}; and likewise for each $X \in \mathcal{C}$ a special map $\varepsilon_X : FGX \to X$ corresponding to 1_{GX}.

These are the components of natural transformations $\eta : 1_{\mathcal{D}} \to GF$ and $\varepsilon : FG \to 1_{\mathcal{C}}$, called the *unit* and *counit* of the adjunction, respectively.

Adjoint Functors

Examples of units/counits in the adjunctions above:

- In the $(-)^{ab} \dashv I$ adjunction, the unit is the projection from a group onto its abelianization; the counit is the identity.
- In the Free group $\dashv$ Forgetful functor adjunction, the unit is the inclusion of the free group's generators (as a function in **Set**); the counit is the obvious map from the free group on the elements of G, onto G.
- In the $(X \times -) \dashv (-)^X$ adjunction the unit is the map $A \to (X \times A)^X$ given by $a \mapsto \lambda x.(x, a)$; the counit $X \times A^X \to A$ is given by the evaluation map.

We may in fact define an adjunction in terms of the unit and counit. We have that $F \dashv G$ if and only if the diagrams of natural transformations

$$\begin{array}{ccc} G & \xrightarrow{\eta G} & GFG \\ & \searrow{=} & \downarrow{G\eta} \\ & & G \end{array} \qquad \begin{array}{ccc} FGF & \xrightarrow{\varepsilon F} & F \\ \uparrow{F\eta} & \nearrow{=} & \\ F & & \end{array}$$

commute.

(Here we're using a notion of composing a natural transformation with a functor that we haven't explained. Don't worry about it.)

Limits & Colimits

Limits & Colimits

Recall our functor Cone_{X_i} from above. Note the following:

- An indexed family of objects in $\mathcal{C}$ is an object of $\mathcal{C}^I$ where I is a discrete category.
- The operation taking $X \in \mathcal{C}$ to the constant family $(X)_{i \in I} \in \mathcal{C}^I$ is (the object part of) a functor $\Delta : \mathcal{C} \to \mathcal{C}^I$.
- $\text{Cone}_{X_i}(Y)$ is the same thing as $\mathcal{C}^I(\Delta Y, X_i)$.

There's nothing special here about discrete I.

Limits & Colimits

Given categories $\mathcal{C}, \mathcal{D}$, there is always a **diagonal functor** $\Delta : \mathcal{C} \to \mathcal{C}^{\mathcal{D}}$ taking $X \in \mathcal{C}$ to the "constantly X" functor $\mathcal{D} \to \mathcal{C}$.

- Given a functor $F : \mathcal{C} \to \mathcal{D}$, a **limit of** F (if it exists) is a representation of $\mathcal{C}^{\mathcal{D}}(\Delta -, F)$.
- A **colimit of** F is a representation of $\mathcal{C}^{\mathcal{D}}(F, \Delta -)$.

A limit P of a functor F, because of representability, is equipped with a universal morphism $\varepsilon_F : \Delta P \to F$.

That is, for each $X \in \mathcal{D}$, a morphism $f_X : P \to FX$ such that for any $g : X \to Y$ in $\mathcal{D}$, $Fg \circ f_X = f_Y$.

- If $\mathcal{D}$ is discrete and P a product of F, $\varepsilon_F \in \mathcal{C}^{\mathcal{D}}(\Delta P, F)$ comprises the projections.

Limits & Colimits

We say $\mathcal{C}$ has **limits of shape** $\mathcal{D}$ (resp. **colimits of shape** $\mathcal{D}$) if $\Delta : \mathcal{C} \to \mathcal{C}^{\mathcal{D}}$ has a right (resp. left) adjoint.

A category is **complete** (resp. **fintely complete**) if $\Delta : \mathcal{C} \to \mathcal{C}^{\mathcal{D}}$ has a right adjoint for every small (resp. finite) $\mathcal{D}$.

A category is **cocomplete** (resp. **fintely cocomplete**) if $\Delta : \mathcal{C} \to \mathcal{C}^{\mathcal{D}}$ has a left adjoint for every small (resp. finite) $\mathcal{D}$.

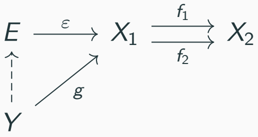
Special Limits & Colimits

Limits of the unique functor in $\mathcal{C}^{\mathbf{0}}$ are **terminal objects** (denoted 1); colimits are **initial objects** (denoted 0).

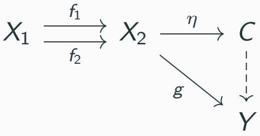
- $X \in \mathcal{C}$ terminal iff for all $Y \in \mathcal{C}$ there is a unique morphism $Y \to X$.
    - Singletons in **Set**
    - The trivial group in **Grp**
    - A top element of a poset
- $X \in \mathcal{C}$ initial iff for all $Y \in \mathcal{C}$ there is a unique morphism $X \to Y$.
    - The empty set in **Set**
    - Also the trivial group in **Grp**
    - Bottom elements of a poset
    - $\mathbb{N}$ gets its inductive properties from being initial in a particular category

Special Limits & Colimits

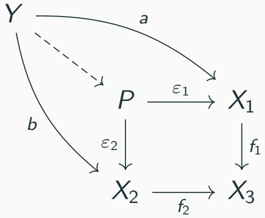
Equalizers are limits of shape $\bullet \rightrightarrows \bullet$.



Coequalizers are colimits for the same diagram.



A limit of shape $\bullet \to \bullet \leftarrow \bullet$ is called a **pullback**.



Note: If f_1 is a monomorphism (c.t. version of injective map), so is ε_2.
Category theoretic version of inverse image.

But. . . Why tho

this was supposed to be a logic seminar

We have talked about the way that representability and adjunctions structure categories. What can we actually do with all this?

- We can state very general desiderata for a category to be a good setting for doing logic (e.g. to make sense of the notion of a structure for some language). For example, model theory works very well in any *Heyting pretopos*.
- Constructions that we can describe as consequences of such representability will port over to other settings (e.g. generalizations of power sets in toposes of different sorts)

Toposes: a preview

In fact, we can do a phenomenal amount of mathematics in a category $\mathcal{E}$ satisfying two reasonable sounding conditions:

- $\mathcal{E}$ is finitely complete. That is

$$\Delta : \mathcal{E} \to \mathcal{E}^{\mathcal{C}}$$

  has a right adjoint for all finite $\mathcal{C}$.
- For all $X \in \mathcal{E}$, the functor $\mathrm{Sub}(- \times X) : \mathcal{E}^{op} \to \mathbf{Set}$ is representable (where $\mathrm{Sub}(Y)$ is the set of subobjects of Y, for a good notion of subobject).

Such a category is called a **topos**; more in the sequel.

Thanks!

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