

Spring 2022

Previous math competitions problems

Wednesday, April 6th, 2022

Mihaela Ifrim

1 Let S be the smallest set of positive integers such that

- (a) 2 is in S ,
- (b) n is in S whenever n^2 is in S , and
- (c) $(n + 5)^2$ is in S whenever n is in S .

Which positive integers are not in S ?

(The set S is “smallest” in the sense that S is contained in any other such set.)

2 Let $Q_0(x) = 1$, $Q_1(x) = x$, and

$$Q_n(x) = \frac{(Q_{n-1}(x))^2 - 1}{Q_{n-2}(x)}$$

for all $n \geq 2$. Show that, whenever n is a positive integer, $Q_n(x)$ is equal to a polynomial with integer coefficients.

3 Let a and b be real numbers with $a < b$, and let f and g be continuous functions from $[a, b]$ to $(0, \infty)$ such that $\int_a^b f(x) dx = \int_a^b g(x) dx$ but $f \neq g$. For every positive integer n , define

$$I_n = \int_a^b \frac{(f(x))^{n+1}}{(g(x))^n} dx.$$

Show that I_1, I_2, I_3, \dots is an increasing sequence with $\lim_{n \rightarrow \infty} I_n = \infty$.

4 A class with $2N$ students took a quiz, on which the possible scores were $0, 1, \dots, 10$. Each of these scores occurred at least once, and the average score was exactly 7.4. Show that the class can be divided into two groups of N students in such a way that the average score for each group was exactly 7.4.

5 Each of the integers from 1 to n is written on a separate card, and then the cards are combined into a deck and shuffled. Three players, A , B , and C , take turns in the order A, B, C, A, \dots choosing one card at random from the deck. (Each card in the deck is equally likely to be chosen.) After a card is chosen, that card and all higher-numbered cards are removed from the deck, and the remaining cards are reshuffled before the next turn. Play continues until one of the three players wins the game by drawing the card numbered 1.

Show that for each of the three players, there are arbitrarily large values of n for which that player has the highest probability among the three players of winning the game.

6 The 30 edges of a regular icosahedron are distinguished by labeling them $1, 2, \dots, 30$. How many different ways are there to paint each edge red, white, or blue such that each of the 20 triangular faces of the icosahedron has two edges of the same color and a third edge of a different color? [Note: the top matter on each exam paper included the Mathematical Association of America, which is itself an icosahedron.]

7 Let L_1 and L_2 be distinct lines in the plane. Prove that L_1 and L_2 intersect if and only if, for every real number $\lambda \neq 0$ and every point P not on L_1 or L_2 , there exist points A_1 on L_1 and A_2 on L_2 such that $\overrightarrow{PA_2} = \lambda \overrightarrow{PA_1}$.

8 Suppose that a positive integer N can be expressed as the sum of k consecutive positive integers

$$N = a + (a + 1) + (a + 2) + \cdots + (a + k - 1)$$

for $k = 2017$ but for no other values of $k > 1$. Considering all positive integers N with this property, what is the smallest positive integer a that occurs in any of these expressions?

9 Suppose that $f(x) = \sum_{i=0}^{\infty} c_i x^i$ is a power series for which each coefficient c_i is 0 or 1. Show that if $f(2/3) = 3/2$, then $f(1/2)$ must be irrational.

10 Evaluate the sum

$$\begin{aligned} & \sum_{k=0}^{\infty} \left(3 \cdot \frac{\ln(4k+2)}{4k+2} - \frac{\ln(4k+3)}{4k+3} - \frac{\ln(4k+4)}{4k+4} - \frac{\ln(4k+5)}{4k+5} \right) \\ &= 3 \cdot \frac{\ln 2}{2} - \frac{\ln 3}{3} - \frac{\ln 4}{4} - \frac{\ln 5}{5} + 3 \cdot \frac{\ln 6}{6} - \frac{\ln 7}{7} \\ & \quad - \frac{\ln 8}{8} - \frac{\ln 9}{9} + 3 \cdot \frac{\ln 10}{10} - \cdots \end{aligned}$$

(As usual, $\ln x$ denotes the natural logarithm of x .)

11 A line in the plane of a triangle T is called an *equalizer* if it divides T into two regions having equal area and equal perimeter. Find positive integers $a > b > c$, with a as small as possible, such that there exists a triangle with side lengths a, b, c that has exactly two distinct equalizers.

12 Find the number of ordered 64-tuples $(x_0, x_1, \dots, x_{63})$ such that x_0, x_1, \dots, x_{63} are distinct elements of $\{1, 2, \dots, 2017\}$ and

$$x_0 + x_1 + 2x_2 + 3x_3 + \cdots + 63x_{63}$$

is divisible by 2017.