Investigating Natural Theories through the Consistency Operator

Graduate Logic Seminar

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Natural Theories

The big question:

Why are "natural" theories usually linearly ordered by consistency strength?

Answer: We don't fully know yet, but there has been some successful attempts.

Example: ordinal analysis.

Ordinal Analysis

One equivalent definition of the proof-theoretic ordinal, the Π_1 ordinal, is defined using iterated consistency statements:

Definition (Π_1 Ordinal)

Fix the base theory EA^+. Define $\mathsf{EA}^+_0 = \mathsf{EA}^+, \mathsf{EA}^+_{\alpha+1} = \mathsf{EA}^+_\alpha \cup \{\mathrm{Con}(\mathsf{EA}^+_\alpha)\}$, and $\mathsf{EA}^+_\lambda = \cup_{\alpha<\lambda} \mathsf{EA}^+_\alpha$ for limit λ.

Then the Π_1 ordinal of any theory T is

$$|T|_{\Pi_1} = \sup\{\alpha | \mathsf{EA}^+_\alpha \subset T\}.$$

The Con Operator

Why was the Con operator used? We might want to say:

Pseudo-claim 1

Con(T) is the weakest (true) statement not determined by T.

Some evidence in support of the claim: incompleteness theorem; low for the jump in provability degrees.

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But it is not true.

Counterexamples: SlowCon; extensional uniform density function.

The Con Operator

So we can improve the pseudo-claim as follows:

Pseudo-claim 2

Con(T) is the weakest (true) *natural* statement not determined by T.

But this remains a pseudo-claim because "naturality" is not well-defined.

Martin's Conjecture

Let's look at another question: what are the "natural" Turing degrees?

Examples: $0, 0', 0'', \ldots, 0^{(\alpha)}, \ldots, \mathcal{O}, \ldots$

Martin's Conjecture

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Examples: $0, 0', 0'', \ldots, 0^{(\alpha)}, \ldots, \mathcal{O}, \ldots$

Instead of asking what the natural Turing degrees are, we can try to classify the "natural" functions on the Turing degrees.

Advantage: easier to "ask for naturality," e.g. uniform/computable/order-preserving/almost everywhere/...

Martin's Conjecture

Here is a rephrased version of Martin's Conjecture.

Conjecture (Martin)

Assume ZF + DC + AD.
(i) If $f : 2^\omega \to 2^\omega$ is degree-invariant, then either f is constant on a cone, or f is above the identity on a cone.
(ii) The relation "$f \leq_T g$ on a cone" prewellorders the set of all degree-invariant functions, with the jump inducing the successor operation.

Parts of the conjecture has been proven when restricted to uniformly degree-invariant/order-preserving functions.

Statement of the Theorem

Goal: Formalize the pseudo-claim by analogy with Martin's Conjecture.

Setting: Work in the Lindenbaum algebra of a base theory (EA by default).

Write $\varphi \vdash \psi$ if EA $\vdash \varphi \to \psi$. Write $[\varphi]$ for the equivalence class of φ. Say φ strictly implies ψ if $\varphi \vdash \psi$ but $\psi \nvdash \varphi$.

Say a sentence φ is true if it is true in the standard model $\mathbb{N}$; consistent if it is consistent with EA, i.e. Con(φ) is true.

A cone has the form $\{\varphi | \varphi \vdash \psi\}$. It is a true cone if ψ is true.

For any function f, say f is extensional if $[\varphi] = [\psi]$ implies $[f(\varphi)] = [f(\psi)]$ (i.e. well-defined on the Lindenbaum algebra); monotonic if $\varphi \vdash \psi$ implies $f(\varphi) \vdash f(\psi)$.

All f are assumed to be computable unless stated otherwise.

Statement of the Theorem

Conjecture

(i) If f is monotonic and above the identity, then either f is the identity on a true cone, or f is above $\mathsf{Con} \wedge \mathsf{Id}$ on a true cone.
(ii) The relation "$f \vdash g$ on a true cone" prewellorders the set of all extensional functions, with the jump inducing the successor operation.

Modifications:

- $\geq_T \to \vdash$;
- Turing jump $\to$ Con, or really $\mathsf{Con} \wedge \mathsf{Id}$;
- Degree-invariant/order-preserving $\to$ extensional/monotonic;
- On a cone $\to$ on a true cone/...
- AD $\to$?

Statement of the Theorem

As a first step, we establish a weakened version of (i).

Theorem

Suppose f is monotonic, and for all consistent φ we have: $f(\varphi)$ strictly implies φ, and $\varphi \wedge \mathrm{Con}(\varphi)$ implies $f(\varphi)$.
Then f and $\mathrm{Con} \wedge \mathrm{Id}$ agrees cofinally, i.e. for every true φ there is a true $\theta \vdash \varphi$ such that $[f(\theta)] = [\theta \wedge \mathrm{Con}(\theta)]$.

Corollary

There is no monotonic function strictly between Id and $\mathrm{Con} \wedge \mathrm{Id}$ (except on $[\perp]$).

Proof of the Theorem

Proof.

Fix a true sentence φ. The following sentence is true:

$$\chi := \forall \xi(\mathsf{Con}(\xi) \to \mathsf{Con}(\xi \wedge \neg f(\xi))).$$

Thus $\psi := \chi \wedge \varphi$ is true. Now define:

$$\theta := \psi \wedge (f(\psi) \to \mathsf{Con}(\psi)).$$

It suffices to show $f(\theta) \vdash \theta \wedge \mathsf{Con}(\theta)$. This is because:

$$f(\theta) \vdash \theta \wedge f(\psi) \vdash \psi \wedge \mathsf{Con}(\psi) \vdash \theta \wedge \mathsf{Con}(\theta),$$

where $\psi \wedge \mathsf{Con}(\psi) \vdash \mathsf{Con}(\theta)$ because: ψ implies χ instantiated at ψ; together with $\mathsf{Con}(\psi)$, this implies $\mathsf{Con}(\theta)$. □

Generalization into the Transfinite

The theorem shows the hierarchy is tight between levels 0 and 1. Now we would like to generalize this result into the transfinite.

Definition (informal)

- $\mathsf{Con}^0(\varphi) = \top$,
- $\mathsf{Con}^{\alpha+1}(\varphi) = \mathsf{Con}(\varphi \wedge \mathsf{Con}^{\alpha}(\varphi))$,
- $\mathsf{Con}^{\lambda}(\varphi) = \forall \alpha < \lambda\, \mathsf{Con}^{\alpha}(\varphi)$ for limit λ.

The rigorous definition requires the fixed point lemma and an elementary presentation (within EA) of the ordinal.

Generalization into the Transfinite

Theorem

Suppose f is monotonic, α is fixed. If for all φ we have: $f(\varphi)$ strictly implies $\varphi \wedge \text{Con}^{\beta}(\varphi)$ for all $\beta < \alpha$, if $[f(\varphi)] \neq [\bot]$; and $\varphi \wedge \text{Con}^{\alpha}(\varphi)$ implies $f(\varphi)$.
Then f and $\text{Con}^{\alpha} \wedge \text{Id}$ agrees cofinally, i.e. for every true φ there is a true $\theta \vdash \varphi$ such that $[f(\theta)] = [\theta \wedge \text{Con}^{\alpha}(\theta)]$.

Corollary

There is no monotonic function strictly above every $\text{Con}^{\beta} \wedge \text{Id}$ (for $\beta < \alpha$) and strictly below $\text{Con}^{\alpha} \wedge \text{Id}$ (except on $[\bot]$).

So the hierarchy is tight.

Generalization into the Transfinite

Proof (Sketch)

We want to replicate the proof for the base case. One important technique is Schmerl's reflexive transfinite induction, i.e.

$$\mathsf{EA} \vdash \forall\alpha(\mathsf{Pr}(\forall\beta < \alpha A(\beta)) \to A(\alpha)) \text{ implies } \mathsf{EA} \vdash \forall\alpha A(\alpha).$$

It follows from Löb's theorem and simplifies induction with Con^α. Here are some important modifications:

- Con^α is monotonic: Induct.
- $\mathsf{Con}^\alpha(\varphi)$ implies $\mathsf{Con}^\alpha(\varphi \wedge \neg f(\varphi))$: might not be true in general, but can relativize to a true cone θ_α defined inductively.

Inevitable Iterates

Say a function is Π^0_k if for all φ, $f(\varphi)$ is Π^0_k.

Theorem

Suppose f is monotonic and Π^0_1, fix α. If f is below $\mathsf{Con}^\alpha \wedge \mathsf{Id}$, then $f \wedge \mathsf{Id}$ agrees with some $\mathsf{Con}^\beta \wedge \mathsf{Id}$ somewhere, for some $\beta \leq \alpha$. Namely, there exists $\beta \leq \alpha$ and some φ such that

$$[\varphi \wedge f(\varphi)] = [\varphi \wedge \mathsf{Con}^\beta(\varphi)] \neq [\perp].$$

If α is finite, we can drop the Π^0_1 assumption.

Inevitable Iterates

Proof (Sketch)

(1) When $\alpha = n$ is finite: suppose towards a contradiction that this is false. Let $g = f \wedge \mathrm{Id}$. Let φ_1 be the conjunction of:
$\forall \xi(\mathrm{Con}(\xi) \to \mathrm{Con}(\xi \wedge \neg g(\xi)))$; and
$\forall k \forall \xi(\mathrm{Con}^{k+1}(\xi) \to \neg \mathrm{Pr}(g(\xi) \leftrightarrow (\xi \wedge \mathrm{Con}^k(\xi))))$.
Extend to a sequence: $\varphi_{k+1} := \varphi_k \wedge (g(\varphi_k) \to \mathrm{Con}^k(\varphi_k))$.
One can show that $\varphi_k \wedge \mathrm{Con}^k(\varphi_k) \vdash \mathrm{Con}^k(\varphi_{k+1})$. Then f and $\mathrm{Con}^n \wedge \mathrm{Id}$ agrees on φ_{n+1}.

(2) For the transfinite case, we need definitions for the limit stages. To this end, we need to have truth predicates uniformly applicable to all φ_α. Requiring f to be Π^0_1 controls the complexity of φ_α, i.e. Π^0_2, giving us a viable truth predicate.

Resolving Part (i)

In fact, part (i) of the Conjecture is true when we restrict to Π_k functions, even if we switch to more general base theories:

Theorem

Let T be an effectively axiomatizable, sound extension of EA, and f be a Π_k monotonic function (for some k). Then either:
(1) $\varphi \vdash f(\varphi)$ on a true cone, or
(2) $\varphi \wedge f(\varphi) \vdash \mathrm{Con}(\varphi)$ on a true cone.

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Proof

Consider the (informal) sentence: $A := \forall x(f(x) \rightarrow \mathrm{Con}(x))$.
If A is false, then $f(\bot)$ is true by extensionality, so the cone $\{\varphi | \varphi \vdash f(\bot)\}$ witnesses (1), using $\bot \rightarrow \varphi$ and monotonicity.
Otherwise, A is true. We claim that $\{\varphi | \varphi \vdash A\}$ witnesses (2). This follows immediately by instantiating A at φ.

Resolving Part (i)

Proof (Continued)

To make the proof fully rigorous, we need to modify our definition of A. First, notice T has the same Σ_1 consequences as true arithmetic. Hence, we have a Σ_1 definition of $G(x, y)$, the graph relation of f, in T. Now define A as:

$$\forall x \forall y ((G(x, y) \wedge \mathrm{True}_{\Pi_k}(y)) \rightarrow \mathrm{Con}(x))$$

The rest of the argument goes through. □

So where did we use a "special" property of Con, in contrast to Con^2, etc.?

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So where did we use a "special" property of Con, in contrast to Con^2, etc.?

Answer: "By extensionality" required us to know $\neg\,\mathrm{Con}(x)$ actually implies $[x] = [\bot]$, i.e. $T \vdash \neg x$. In this sense, Con is really universal.

Next Steps

Now we have several possible directions to proceed.

- Prove Part (ii) by showing cofinal agreement implies true cone equivalence.
- Prove Part (ii) by generalizing the proof for Part (i).
- Prove Part (i) for less computable functions.
- Prove Part (i) for non-Π_k functions.

However, all but the last approach fail. First notice that true cone equivalence implies cofinal agreement; that the intersection of a cofinal set with a true cone is still cofinal; and that different iterates of Con agree nowhere.

Negative Results

All results below apply to any effectively axiomatizable, sound base theory T extending EA except the first one (only stated for EA), though the proof of the first likely also works for general T.

Theorem

There is a degree-invariant cofinal c.e. set containing no true cone.

Theorem

For any α, there is a computable f which is Π_1 and monotonic, but $f \wedge \mathsf{Id}$ agrees cofinally with both $\mathsf{Con} \wedge \mathsf{Id}$ and $\mathsf{Con}^\alpha \wedge \mathsf{Id}$.

Theorem

There is a limit computable (i.e. $\leq_T 0'$) f which is Π_1 and monotonic, but $f \wedge \mathsf{Id}$ agrees cofinally with both Id and $\mathsf{Con} \wedge \mathsf{Id}$.

As a corollary, cofinal agreement is not transitive.

Further Questions

- Generalizing Part (i) to non-Π_k functions.
- Analog of AD?
- An alternative notion stronger than cofinal agreement yet weaker than true cone equivalence?

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Thank you for listening!