Cotangent Complex

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Outline

- DGA and Semifree Resolution
- Relative Differential
- Cotangent Complex: Definition and Properties
- Applications

Motivation for dg-algebra

We need our resolution to have an algebra structure. This idea can help us simplify a lot of proofs.

Proposition

$\mathrm{Tor}_k^\bullet(A, B)$ has an k-algebra structure if A, B are both k-algebras.

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If we use only modules to resolve algebras, we need to worry about lifting the multiplication maps, associativity, and commutativity.

Differential Graded Algebra

Definition ((C)DGA)

A *differential graded k-algebra* is a triple (A, m_1, m_2) where

- A is a graded k-module;
- $m_1 \colon A \to A$ has degree 1 and satisfies $m_1^2 = 0$;
- $m_2 \colon A^{\otimes 2} \to A$ has degree 0 and obeys the Leibniz rule

$$m_1\big(m_2(a \otimes b)\big) = m_2\big(m_1(a) \otimes b\big) + (-1)^{|a|}\, m_2\big(a \otimes m_1(b)\big);$$

- if A is *graded-commutative*, then

$$m_2(a \otimes b) = (-1)^{|a||b|}\, m_2(b \otimes a).$$

Semifree resolution

Definition

A *semifree* differential graded (dg) algebra $F = (F^\bullet, d)$ over a field k is a filtered dg algebra

$$k = F^{(-1)} \subset F^{(0)} \subset F^{(1)} \subset \cdots \subset \bigcup_{i \geq 0} F^{(i)} = F$$

such that

1. $F^{(i)}$ is a *free* (i.e. symmetric or exterior tensor algebra or a combination of both) on some graded vector space $V^{(i)}$ over $F^{(i-1)}$, and
2. the differential satisfies $d\big(V^{(i)}\big) \subset F^{(i-1)}$ for every i.

A *semifree resolution* of a dg-algebra A is a surjective quasi-isomorphism $F \xrightarrow{\simeq} A$ with F semifree.

Remark: We can also define a relative version of semifree dg algebra, see Def 4.1. of [Man11].

Semifree resolution

Example

Compute the semifree resolution of $k[x, y]/(xy)$.

Example

Step 1: Add one generator

$$F_1 \;=\; \big(k[x,y] \;\otimes_k\; k\langle e\rangle,\ |e|=1,\ de=xy\big).$$

As graded algebras $F_1 \cong k[x,y] \otimes \Lambda(e)$, so it is free on the single generator e.

Step 2: Compute homology

- *Degree 0.* Boundaries form the ideal $(xy) \subset k[x,y]$, so

$$H^0(F_1) \;\cong\; k[x,y]/(xy) \;=\; A.$$

- *Degree 1.* A cycle is $f(x,y)\,e$ with $d(f\,e) = f\,xy = 0$. Because $k[x,y]$ is a domain, $f=0$; hence $H^1(F_1)=0$.
- *Higher degrees.* None exist, so $H^{\geq 2}(F_1)=0$.

Therefore all higher homology groups vanish, and F_1 is already a semifree resolution of A.

Semifree resolution

Theorem 4.7. of [Man11]

Any K-dg-algebra admits a semifree resolution.

Tensor Product of DGAs

Definition

Let $B \to A$ and $B \to C$ be morphisms of differential graded k-algebras. The *tensor product*

$$A \otimes_B C := \frac{A \otimes_k C}{\langle\, a \otimes bc \;-\; ab \otimes c \mid a \in A,\ b \in B,\ c \in C \,\rangle}$$

is again a differential graded k-algebra, with structure maps:

- **Grading:** $|a \otimes c| \;=\; |a| + |c|$.
- **Differential (Leibniz rule):** $m_1(a \otimes c) \;=\; m_{1,A}(a) \otimes c \;+\; (-1)^{|a|}\, a \otimes m_{1,C}(c)$.
- **Multiplication:** $m_2\big((a_1 \otimes c_1) \otimes (a_2 \otimes c_2)\big) \;=\; (-1)^{|a_2||c_1|}\, m_{2,A}(a_1 \otimes a_2) \otimes m_{2,C}(c_1 \otimes c_2)$.

Derivations and Kahler Differentials

Definition

The k-dg-module of derivations is a dg-submodule of $\mathrm{Hom}^\bullet(A,M)$ whose degree n piece is given by

$$\mathrm{Der}^n_B(A,M) = \{\phi \in \mathrm{Hom}^n_B | \phi(ab) = \phi(a)b + (-1)^{|a|} a\phi(b), \phi(B) = 0\}$$

Derivations and Kähler differentials

Proposition

Let $B \to A$ be a morphism of differential graded (dg) algebras. There exists an A-dg-module $\Omega_{A/B}$ together with a closed derivation

$$\delta : A \longrightarrow \Omega_{A/B}, \qquad |\delta| = 0,$$

such that for every A-dg-module M the map

$$\mathrm{Hom}^*_A(\Omega_{A/B}, M) \xrightarrow{\circ\,\delta} \mathrm{Der}^*_B(A, M)$$

is an isomorphism.

Derivations and Kähler differentials

Sketch of Proof: Consider the graded A-dg-module

$$F_A = \bigoplus_{x \in A \text{ homogeneous}} A\delta x$$

with differential $d(a\delta b) = da\delta b + (-1)^{|a|} a\delta(db)$. Let I be the homogeneous submodule generated by

$$\delta(B),\ \delta(x+y) - \delta x - \delta y,$$

$$\delta(xy) - \delta(x)y - (-1)^{|x|} x\delta(y)$$

Construct $\Omega_{A/B}$ by taking quotient F_A/I. Show that $\circ\delta$ is a morphism of A-dg-modules, and that it's surjective and injective.

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Remark: There are other constructions. Consider the kernel I of $A \otimes_B A \to A$. Let $\Omega_{A/B}$ be I/I^2.

Cotangent Complex

Proposition

Suppose we have a diagram of K-dg-algebras

$$\begin{array}{ccccc} R & \xrightarrow{f} & S & \xleftarrow{g} & R \\ & {\scriptstyle p}\searrow & \downarrow & \swarrow{\scriptstyle p} & \\ & & A & & \end{array}$$

If there exists a homotopy (See [Man11]) between f and g, constant on A, then the induced morphisms of A-dg-modules

$$f,\, g : \ \Omega_{R/K} \otimes_R A \longrightarrow \Omega_{S/K} \otimes_S A$$

are homotopic.

Cotangent Complex

Definition (Cotangent Complex)

Let $R \to A$ be a K-semifree resolution. The A-dg-module

$$\mathbb{L}_{A/K} \;=\; \Omega_{R/K} \otimes_R A$$

is called the *relative cotangent complex* of A over K. By the previous proposition, the homotopy class of $\mathbb{L}_{A/K}$ is independent of the chosen resolution.

For every A-dg-module M define

$$T^i(A, M) \;=\; H^i\big(\mathrm{Hom}^*_A(\mathbb{L}_{A/K}, M)\big) \;=\; \mathrm{Ext}^i_A(\mathbb{L}_{A/K}, M),$$

$$T_i(A, M) \;=\; H_i\big(\mathbb{L}_{A/K} \otimes_A M\big) \;=\; \mathrm{Tor}^A_i(\mathbb{L}_{A/K}, M).$$

These are called, respectively, the *cotangent* and *tangent cohomology* of the morphism $K \to A$ with coefficients in M.

Example: $\mathbb{L}_{k[x,y]/(xy)}$

1. **Semifree cdga resolution**

$$R = k[x,y,T], \quad \deg x = \deg y = 0,\ \deg T = -1,$$
$$d(x) = d(y) = 0,\ d(T) = xy,$$

with augmentation $\epsilon : R \twoheadrightarrow A = k[x,y]/(xy)$ sending $T \mapsto 0$.

2. **Kähler differentials of R**

$$\Omega^1_{R/k} = R\,dx \ \oplus\ R\,dy \ \oplus\ R\,dT, \quad d(dT) = y\,dx + x\,dy.$$

This is a two-term complex (note that the actual degree is different because R is a dg-algebra)

$$R\,dT \xrightarrow{(y,-x)} R\,dx \oplus R\,dy, \qquad \text{degrees } (-1 \to 0).$$

3. **Tensoring down to A**

$$L_{A/k} \ \simeq\ \Omega^1_{R/k} \otimes_R A = \left[A \xrightarrow{(y,-x)} A^{\oplus 2}\right], \quad \text{degrees } [-1,0].$$

Hence $H^{-1}(L_{A/k}) \cong A/(x,y) \cong k$ and $\Omega^1_{A/k} = H^0(L_{A/k}) = \dfrac{A\,dx \oplus A\,dy}{\langle y\,dx + x\,dy \rangle}$.

Properties of the Cotangent Complex

Proposition

If $R \to A$ is a semifree resolution, then

$$H^i(\mathrm{Der}^\bullet(R,R)) \cong \mathrm{Ext}^i(L_A, A)$$

Proposition

If $X \xrightarrow{f} Y \to Z$ is a morphism of schemes, then we have an exact triangle in $D^-_{Qcoh}(\mathcal{O}_X)$

$$f^* L_{Y/Z} \to L_{X/Z} \to L_{X/Y} \to f^* L_{Y/Z}[1].$$

Applications to Deformation Theory

Square-zero extensions (See [Sta24])

We start with a surjective ring map $A' \twoheadrightarrow A$ whose kernel I satisfies $I^2 = 0$. Suppose further that we have

$A \longrightarrow B$, N a B-module, $c : I \longrightarrow N$ an A-module map.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N & \longrightarrow & ? & \longrightarrow & B & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle c} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & 0
\end{array}$$

Goal. Find an A'-algebra $B' \twoheadrightarrow B$ with square-zero kernel identified with N such that $A' \to B'$ induces the map c. We call any such B' a *solution* to the diagram and ask how many (up to iso) exist.

Lemma – obstruction and classification

In the situation above:

(1) There is a canonical class $\xi \in \mathrm{Ext}^2_B(L_{B/A}, N)$ whose vanishing is *necessary and sufficient* for the existence of a solution B'.

(2) If a solution exists, the set of isomorphism classes of solutions is a principal homogeneous space under $\mathrm{Ext}^1_B(L_{B/A}, N)$.

(3) For any fixed solution B', the automorphism group of B' (within the diagram) is canonically $\mathrm{Ext}^0_B(L_{B/A}, N)$.

Remark: (2) and (3) can be reinterpreted as the following:

Proposition

Let $\mathcal{D}$ be the category of solutions to the above extension problem, with morphisms being isomorphisms, then

$$\mathcal{D} \cong [\mathrm{Ext}^1_B(L_{B/A}, N)/\mathrm{Ext}^0(L_{B/A}, N)]$$

[Alp23, §C.3] — Extensions by square-zero ideals

Definition

Let $X \to S$ be a morphism of schemes and let J be a quasi-coherent $\mathcal{O}_X$-module. An *extension of X by J (relative to S)* is a short exact sequence of sheaves of rings

$$0 \longrightarrow J \longrightarrow \mathcal{O}_{X'} \longrightarrow \mathcal{O}_X \longrightarrow 0,$$

where $X \hookrightarrow X'$ is the closed immersion defined by the ideal sheaf $J \subseteq \mathcal{O}_{X'}$ with $J^2 = 0$. (The condition $J^2 = 0$ forces J to be an $\mathcal{O}_X$-module.)

The *trivial extension* is $X[J] := (X, \mathcal{O}_X \oplus J)$ with ring structure $(j_1, j_2)^2 = 0$.

A *morphism of extensions* is a morphism of short exact sequences which is the identity on J and $\mathcal{O}_X$. We write $\mathbf{Exal}_S(X, J)$ for the category of such extensions and $\mathrm{Exal}_S(X, J)$ for its set of isomorphism classes.

Geometric Interpretation

Remark: Geometrically an extension is a commutative diagram of schemes

$$\begin{array}{ccc} X & \hookrightarrow & X' \\ \downarrow & \swarrow & \\ \operatorname{Spec} R & & \end{array}$$

where the top arrow is a closed immersion with square-zero kernel J.

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