Cotangent Complex

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Outline

- DGA and Semifree Resolution
- Relative Differential
- Cotangent Complex: Definition and Properties
- Applications

Motivation for dg-algebra

We need our resolution to have an algebra structure. This idea can help us simplify a lot of proofs.

Proposition

 $\operatorname{Tor}_k^{\bullet}(A,B)$ has an k-algebra structure if A, B are both k-algebras.

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If we use only modules to resolve algebras, we need to worry about lifting the multiplication maps, associativity, and commutativity.

Differential Graded Algebra

Definition ((C)DGA)

A differential graded k-algebra is a triple (A, m_1, m_2) where

- A is a graded k-module;
- $m_1: A \to A$ has degree 1 and satisfies $m_1^2 = 0$;
- $m_2 \colon A^{\otimes 2} \to A$ has degree 0 and obeys the Leibniz rule

$$m_1(m_2(a \otimes b)) = m_2(m_1(a) \otimes b) + (-1)^{|a|} m_2(a \otimes m_1(b));$$

• if A is graded-commutative, then

$$m_2(a \otimes b) = (-1)^{|a||b|} m_2(b \otimes a).$$

Semifree resolution

Definition

A semifree differential graded (dg) algebra $F=(F^{\bullet},d)$ over a field k is a filtered dg algebra

$$k = F^{(-1)} \subset F^{(0)} \subset F^{(1)} \subset \cdots \subset \bigcup_{i>0} F^{(i)} = F$$

such that

- $F^{(i)}$ is a free (i.e. symmetric or exterior tensor algebra or a combination of both) on some graded vector space $V^{(i)}$ over $F^{(i-1)}$, and
- 2 the differential satisfies $d(V^{(i)}) \subset F^{(i-1)}$ for every i.

A semifree resolution of a dg-algebra A is a surjective quasi-isomorphism $F \xrightarrow{\cong} A$ with F semifree.

Remark: We can also define a relative version of semifree dg algebra, see Def 4.1. of [Man11].

Semifree resolution

Example

Compute the semifree resolution of k[x,y]/(xy).

Example

Step 1: Add one generator

$$F_1 = (k[x,y] \otimes_k k\langle e \rangle, |e| = 1, de = xy).$$

As graded algebras $F_1 \cong k[x,y] \otimes \Lambda(e)$, so it is free on the single generator e.

Step 2: Compute homology

• Degree 0. Boundaries form the ideal $(xy) \subset k[x,y]$, so

$$H^0(F_1) \cong k[x,y]/(xy) = A.$$

- Degree 1. A cycle is f(x,y)e with d(fe) = fxy = 0. Because k[x,y] is a domain, f=0; hence $H^1(F_1)=0$.
- Higher degrees. None exist, so $H^{\geq 2}(F_1) = 0$.

Therefore all higher homology groups vanish, and F_1 is already a semifree resolution of A.

Semifree resolution

Theorem 4.7. of [Man11]

Any K-dg-algebra admits a semifree resolution.

Tensor Product of DGAs

Definition

Let $B \to A$ and $B \to C$ be morphisms of differential graded k-algebras. The $tensor\ product$

$$A \otimes_B C := \frac{A \otimes_k C}{\langle a \otimes bc - ab \otimes c \mid a \in A, b \in B, c \in C \rangle}$$

is again a differential graded k-algebra, with structure maps:

- Grading: $|a \otimes c| = |a| + |c|$.
- Differential (Leibniz rule): $m_1(a \otimes c) = m_{1,A}(a) \otimes c + (-1)^{|a|} a \otimes m_{1,C}(c).$
- Multiplication: $m_2((a_1 \otimes c_1) \otimes (a_2 \otimes c_2)) = (-1)^{|a_2||c_1|} m_{2,A}(a_1 \otimes a_2) \otimes m_{2,C}(c_1 \otimes c_2).$

Derivations and Kahler Differentials

Definition

The k-dg-module of derivations is a dg-submodule of $\mathrm{Hom}^{\bullet}(A,M)$ whose degree n piece is given by

$$\operatorname{Der}_{B}^{n}(A, M) = \{ \phi \in \operatorname{Hom}_{B}^{n} | \phi(ab) = \phi(a)b + (-1)^{|a|} a\phi(b), \phi(B) = 0 \}$$

Derivations and Kähler differentials

Proposition

Let $B \to A$ be a morphism of differential graded (dg) algebras. There exists an A-dg-module $\Omega_{A/B}$ together with a closed derivation

$$\delta: A \longrightarrow \Omega_{A/B}, \qquad |\delta| = 0,$$

such that for every A-dg-module M the map

$$\operatorname{Hom}_{A}^{*}(\Omega_{A/B}, M) \xrightarrow{\circ \delta} \operatorname{Der}_{B}^{*}(A, M)$$

is an isomorphism.

Derivations and Kähler differentials

Sketch of Proof: Consider the graded A-dg-module

$$F_A = \bigoplus_{x \in A \text{ homogeneous}} A\delta x$$

with differential $d(a\delta b)=da\delta b+(-1)^{|a|}a\delta(db)$. Let I be the homogeneous submodule generated by

$$\delta(B), \ \delta(x+y) - \delta x - \delta y,$$

$$\delta(xy) - \delta(x)y - (-1)^{|x|}x\delta(y)$$

Construct $\Omega_{A/B}$ by taking quotient F_A/I . Show that $\circ \delta$ is a morphism of A-dg-modules, and that it's surjective and injective.

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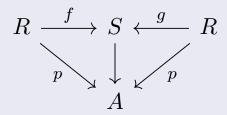
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Remark: There are other constructions. Consider the kernel I of $A \otimes_B A \to A$. Let $\Omega_{A/B}$ be I/I^2 .

Cotangent Complex

Proposition

Suppose we have a diagram of K-dg-algebras



If there exists a homotopy (See [Man11]) between f and g, constant on A, then the induced morphisms of A-dg-modules

$$f, g: \Omega_{R/K} \otimes_R A \longrightarrow \Omega_{S/K} \otimes_S A$$

are homotopic.

Cotangent Complex

Definition (Cotangent Complex)

Let $R \to A$ be a K-semifree resolution. The A-dg-module

$$\mathbb{L}_{A/K} = \Omega_{R/K} \otimes_R A$$

is called the *relative cotangent complex* of A over K. By the previous proposition, the homotopy class of $\mathbb{L}_{A/K}$ is independent of the chosen resolution.

For every A-dg-module M define

$$T^{i}(A, M) = H^{i}(\operatorname{Hom}_{A}^{*}(\mathbb{L}_{A/K}, M)) = \operatorname{Ext}_{A}^{i}(\mathbb{L}_{A/K}, M),$$

$$T_i(A, M) = H_i(\mathbb{L}_{A/K} \otimes_A M) = \operatorname{Tor}_i^A(\mathbb{L}_{A/K}, M).$$

These are called, respectively, the *cotangent* and *tangent* cohomology of the morphism $K \to A$ with coefficients in M.

Example: $\mathbb{L}_{k[x,y]/(xy)}$

Semifree cdga resolution

$$R = k[x, y, T], \quad \deg x = \deg y = 0, \ \deg T = -1,$$

 $d(x) = d(y) = 0, \ d(T) = xy,$

with augmentation $\epsilon:R \twoheadrightarrow A=k[x,y]/(xy)$ sending $T\mapsto 0$.

 $oldsymbol{2}$ Kähler differentials of R

$$\Omega^1_{R/k} = R dx \oplus R dy \oplus R dT, \quad d(dT) = y dx + x dy.$$

This is a two-term complex (note that the actual degree is different because R is a dg-algebra)

$$R dT \xrightarrow{(y,-x)} R dx \oplus R dy$$
, degrees $(-1 \to 0)$.

3 Tensoring down to A

$$L_{A/k} \; \simeq \; \Omega^1_{R/k} \otimes_R A = \left[A \xrightarrow{(y,-x)} A^{\oplus 2} \right], \quad \text{degrees } [-1,0].$$
 Hence $H^{-1}(L_{A/k}) \cong A/(x,y) \cong k$ and $\Omega^1_{A/k} = H^0(L_{A/k}) = \frac{A\,dx \oplus A\,dy}{\langle y\,dx + x\,dy \rangle}.$

Properties of the Cotangent Complex

Proposition

If $R \to A$ is a semifree resolution, then

$$H^i(\mathrm{Der}^{\bullet}(R,R)) \cong \mathrm{Ext}^i(L_A,A)$$

Proposition

If $X \xrightarrow{f} Y \to Z$ is a morphism of schemes, then we have an exact triangle in $D^-_{Qcoh}(\mathcal{O}_X)$

$$f^*L_{Y/Z} \to L_{X/Z} \to L_{X/Y} \to f^*L_{Y/Z}[1].$$



Square-zero extensions (See [Sta24])

We start with a surjective ring map A' woheadrightarrow A whose kernel I satisfies $I^2 = 0$. Suppose further that we have

Goal. Find an A'-algebra B' B with square-zero kernel identified with N such that A' B' induces the map c. We call any such B' a *solution* to the diagram and ask how many (up to iso) exist.

Lemma – obstruction and classification

In the situation above:

- (1) There is a canonical class $\xi \in \operatorname{Ext}^2_B(L_{B/A}, N)$ whose vanishing is *necessary and sufficient* for the existence of a solution B'.
- (2) If a solution exists, the set of isomorphism classes of solutions is a principal homogeneous space under $\operatorname{Ext}^1_B(L_{B/A}, N)$.
- (3) For any fixed solution B', the automorphism group of B' (within the diagram) is canonically $\operatorname{Ext}_B^0(L_{B/A}, N)$.

Remark: (2) and (3) can be reinterpreted as the following:

Proposition

Let \mathcal{D} be the category of solutions to the above extension problem, with morphisms being isomorphisms, then

$$\mathcal{D} \cong [\operatorname{Ext}_B^1(L_{B/A}, N) / \operatorname{Ext}^0(L_{B/A}, N)]$$

[Alp23, §C.3] — Extensions by square-zero ideals

Definition

Let $X \to S$ be a morphism of schemes and let J be a quasi-coherent \mathcal{O}_X -module. An extension of X by J (relative to S) is a short exact sequence of sheaves of rings

$$0 \longrightarrow J \longrightarrow \mathcal{O}_{X'} \longrightarrow \mathcal{O}_X \longrightarrow 0$$
,

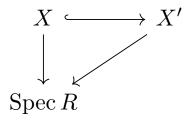
where $X \hookrightarrow X'$ is the closed immersion defined by the ideal sheaf $J \subseteq \mathcal{O}_{X'}$ with $J^2 = 0$. (The condition $J^2 = 0$ forces J to be an \mathcal{O}_X -module.)

The trivial extension is $X[J] := (X, \mathcal{O}_X \oplus J)$ with ring structure $(j_1, j_2)^2 = 0$.

A morphism of extensions is a morphism of short exact sequences which is the identity on J and \mathcal{O}_X . We write $\mathbf{Exal}_S(X,J)$ for the category of such extensions and $\mathrm{Exal}_S(X,J)$ for its set of isomorphism classes.

Geometric Interpretation

Remark: Geometrically an extension is a commutative diagram of schemes



where the top arrow is a closed immersion with square-zero kernel J.

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