Math 763.
Correspondence between ideals and algebraic sets.

In this note, we list the properties of the operations $I$ and $V$. Recall that

- For a subset $S \subset k[x_1, \ldots, x_n]$, $V(S)$ is the common zero locus of $S$;
- For a subset $X \subset k^n$, $I(X)$ is the set of all functions vanishing on $X$.

The following properties are supposed to be obvious; if they are not, treat them as exercises. (And try not to use the Nullstellensatz unless absolutely necessary.)

1. $I(X) = \bigcap_{a \in X} m_a$, where $m_a = (x-a_1, \ldots, x-a_n)$ for $a = (a_1, \ldots, a_n) \in k^n$. (Reformulation of the definition.)
2. $V(S) = \{ a \in k^n : m_a \supset S \}$. (Reformulation of the definition.)
3. $I(X)$ is a radical ideal.
4. $V(S)$ is an algebraic set (by definition).
5. If $S_1 \subset S_2$, then $V(S_1) \supset V(S_2)$.
6. If $X_1 \subset X_2$, then $I(X_1) \supset I(X_2)$.
7. $I(\emptyset) = k[x_1, \ldots, x_n]$, $I(k^n) = 0$.
8. $V(\emptyset) = V(\emptyset) = k^n$, $V(\{1\}) = V(\{\}) = \emptyset$.
9. $V(I(X))$ is the smallest algebraic set containing $X$.
10. $I(V(S)) = \text{ideal generated by } S$.
11. $V(S_1 \cup S_2) = V(S_1) \cap V(S_2)$, more generally, $V(\bigcup_{\alpha} S_\alpha) = \bigcap_{\alpha} V(S_\alpha)$.
12. $I(X_1 \cap X_2) = I(X_1) \cup I(X_2)$, more generally, $I(\bigcap_{\alpha} X_\alpha) = \bigcup_{\alpha} I(X_\alpha)$.
13. $V(S_1 \cdot S_2) = V(S_1) \cup V(S_2)$, where $S_1 \cdot S_2 := \{ f_1 f_2 : f_1 \in S_1, f_2 \in S_2 \}$.

Now let us restrict the operation $V$ to (not necessarily radical) ideals.

14. $V(I_1 + I_2) = V(I_1) \cap V(I_2)$;
15. More generally, $V(\bigcup_{\alpha} I_\alpha) = \bigcap_{\alpha} V(I_\alpha)$;
16. $V(I_1 \cdot I_2) = V(I_1 \cap I_2) = V(I_1) \cup V(I_2)$.

Note that even if $I_1 = \sqrt{I_1}$ and $I_2 = \sqrt{I_2}$, it is not necessarily true that $\sqrt{I_1 + I_2} = (I_1 + I_2)$ or $\sqrt{I_1 \cdot I_2} = (I_1 \cdot I_2)$, but it is true that $\sqrt{I_1 \cap I_2} = I_1 \cap I_2$. We then see that $\sqrt{I_1 \cdot I_2} = I_1 \cap I_2$.

Finally, if we restrict $I$ and $V$ to algebraic subsets of $k^n$ and radical ideals of $k[x_1, \ldots, x_n]$ respectively, they become order-reversing bijections.