ALGEBRA QUALIFYING EXAM, JANUARY 2015

(1) This problem concerns expressing groups as unions of proper subgroups.
  (a) Show that no group is the union of two proper subgroups.
  (b) Show that $\mathbb{Z}$ is not the union of any number of proper subgroups.
  (c) For which n is $\mathbb{Z}^n$ the union of finitely many proper subgroups? What is the minimal number of such subgroups as a function of n?

(2) Let L/K be a finite extension of fields whose degree $p = [L : K]$ is a prime number. Let σ be an automorphism of L over K (that is, an element of the Galois group $\mathrm{Gal}(L/K)$), $\sigma \neq id$. In particular, σ is a linear operator on the p-dimensional K-vector space L. Suppose σ is diagonalizable over K (that is, there is a basis of L as K-vector space consisting of eigenvectors for σ). Show that L is the splitting field of the polynomial $x^p - a$ for some $a \in K$.

(3) Let V be a subspace of $\mathbb{C}^n$ spanned by the basis vectors $e_1, \ldots, e_m$, and let E be the set of matrices A in $M_n(\mathbb{C})$ such that $AV \subset V$.
  (a) Show that E is an algebra.
  (b) Compute all the two-sided ideals of E.

(4) Let F be a field, and let V be a finite-dimensional vector space over F that has dimension n ($1 \leqslant n < \infty$). Let q be a nonzero element of F such that $q^i \neq 1$ for $1 \leqslant i \leqslant n$. Suppose given F-linear transformations $X : V \to V$ and $Y : V \to V$ such that $XY = qYX$. Show that XY is nilpotent.

(5) Let $R = k[x]$, for k an algebraically closed field, and let M be a finitely generated R-module. Define the rank of M to be the dimension of the $k(x)$-vector space $M_{(0)}$. (Here (0) is the prime ideal in R, and the subscript denotes localization.)
  (a) Prove that the rank of M is finite.
  (b) If the rank of M is zero, prove that there exists a polynomial $f \in R$ such that $M_f = 0$. (Recall that M_f denotes the appropriate localization.)
  (c) Consider the function $f : k \to \mathbb{Z}$ given by

$$f(a) = \dim_k M \otimes_R R/(x-a).$$

  Prove that for all but finitely many $a \in k$ we have $f(a) = \operatorname{rank} M$.