ALGEBRA QUALIFYING EXAM, JANUARY 2016

1. Let K be a field, and let $f(x) \in K[x]$ be an irreducible polynomial. Suppose that the splitting field L of $f(x)$ is a Galois extension of K (that is, f is separable). Let $\alpha \in L$ be one of the roots of f.

(a) Show that if $\mathrm{Gal}(L/K)$ is an abelian group, then $L = K(\alpha)$.

(b) Is the converse statement true? That is, suppose $L = K(\alpha)$; must $\mathrm{Gal}(L/K)$ be abelian?

2. Let A be a real skew-symmetric $n \times n$ matrix such that $\mathrm{im}(A) = \ker(A)$. In particular, $\dim(\mathrm{im}(A)) = n/2$, so n must be even.

(a) Let V be an $n/2$-dimensional subspace of $\mathbb{R}^n$. Define a bilinear form $(,)$ on V by

$$(v, w) = \langle v, Aw \rangle;$$

here $\langle , \rangle$ is the standard Euclidean product on $\mathbb{R}^n$. Show that $(,)$ is a skew-symmetric form on V.

(b) Show that V can be chosen so that the above form $(,)$ is non-degenerate.

(c) Show that n must be divisible by 4. If you use some facts about non-degenerate skew-symmetric forms, sketch their proofs.

3. Let $\phi : A \to B$ be a homomorphism of commutative rings. Recall that ϕ is said to be *integral* if every element of $b \in B$ is integral over $\phi(A)$, and that ϕ is *finite* if B is a finitely generated A-module.

(a) Give an example of a map of commutative rings $A \to B$ that is integral but not finite.

(b) Prove the Lying Over Theorem: Let $\phi : A \to B$ be an integral map of commutative rings. If $\mathfrak{p} \subset A$ is any prime ideal, then there exists a prime ideal $\mathfrak{q} \subset B$ such that $\phi^{-1}(\mathfrak{q}) = \mathfrak{p}$.

4. Let D_k be the dihedral group of order $2k$, where $k \geq 3$.

(a) Show that the number of automorphisms of the group D_k is equal to $k \cdot \phi(k)$. Here ϕ is Euler's ϕ-function.

(b) The automorphisms of D_k form a group; let us denote it by $\mathrm{Aut}(D_k)$. What is the structure of $\mathrm{Aut}(D_k)$? Describe the group as explicitly as you can.

5. Consider the group $\mathrm{GL}_n(\mathbb{Q})$ of invertible $n \times n$ matrices with rational coefficients. Suppose $G \subset \mathrm{GL}_n(\mathbb{Q})$ is a finite subgroup. Prove that every prime factor p of the order $|G|$ satisfies $p \leq n + 1$.