September 26: Reductive Groups

I. Definitions and Examples
II. Root Data; Existence and Isogeny Theorem
III. Representations of Reductive Groups.

I. Definitions and Examples

Definition An algebraic group G over a field k is a scheme of finite type $/k$ together with a morphism
$m: G \times G \to G$ such that
i) $\exists$ maps $e: \operatorname{Spec} k \to G$, $i: G \to G$ such that
ii) the usual diagrams defining a group all commute.

"usual diagrams" e.g. Associativity:

$$\begin{tikzcd} G\times G\times G \arrow[r, "id\times m"] \arrow[d, "m\times id"'] & G\times G \arrow[d, "m"] \\ G\times G \arrow[r, "m"'] & G \end{tikzcd}$$

Definition (G,m) is an affine algebraic group if G is an affine scheme.

e.g. the multiplicative group $\mathbb{G}_m = \operatorname{Spec}(k[x,x^{-1}])$.

(G,m) is smooth (resp. connected) if G is a smooth (resp. connected) scheme.

G is smooth iff some point $\operatorname{spec}(k) \to G$ is smooth.
All alg groups are smooth if char. $k = 0$.

G is connected iff: $\begin{cases} G \text{ is geometrically connected, i.e. } G \times_k \bar{k} \text{ is connected.} \\ G \text{ is irreducible.} \end{cases}$

Definition A group G/k is reductive if G is smooth, connected, and has trivial unipotent radical.
The unipotent radical of G is the maximal normal unipotent subgroup of G.
A ~~unipotent~~ group $U/_k$ is unipotent if all homomorphisms $G \to GL(V)$ stabilize a vector of V.

The previous definitions require some proof: existence of a well-defined maximal unipotent subgroup is nontrivial.

~~GL~~ $GL(V)$ where $V \cong \mathbb{A}^n_k$ is defined by $k[x_1, \dots, x_n; \det^{-1}]$.

An <u>algebraic subgroup</u> of G is a closed subscheme H of G together with the morphism $m|_{H \times H}$.

<u>Examples</u> • Classical groups GL_n, SL_n, SO_n, etc. are reductive.

- PGL_n is reductive.
- $\mathbb{G}_a$ has nontrivial unipotent radical $(= \mathbb{G}_a)$.
- Upper triangular matrices with diagonal of 1's ~~has~~ has unipotent radical equal to itself.

<u>Motivation</u> for considering reductive groups

- In characteristic zero, reductive = <u>linearly reductive</u>: Rep_G over k is semisimple.
- No longer true in positive characteristic (e.g. $GL_p \curvearrowright V^{\otimes p}$ cannot split)

<u>Definition</u> A <u>split torus</u> over k is an algebraic group isomorphic to $\mathbb{G}_m^{\times n}$.

A <u>torus</u> is an algebraic group T ~~s~~ such that $T \times_k \bar{k} \simeq \mathbb{G}^{\times n}_{m\,\bar{k}}$.

Example: $G/\mathbb{R}$: $\mathrm{Spec}\left(\frac{\mathbb{R}[x,y]}{\langle x^2+y^2-1 \rangle}\right)$ is not split.

<u>Definition</u> A reductive group G is <u>splittable</u> if it contains a split torus.

A <u>split reductive group</u> is a pair (G, T) where G is splittable and T is a fixed split torus which is <u>maximal</u>.

<u>Examples</u> • $(GL_2, \{\text{diagonal matrices}\})$ is a split reductive group.

- $(SL_2, \{\text{diagonal matrices}\})$ is a split reductive group.
- $(SO_3, \{\text{matrices fixing an axis of } SO_3 \curvearrowright k^3\}$.
- (Compact group $/\mathbb{R}$, maximal torus) is reductive but not splittable.

There is a nice way to consider $\mathrm{Lie}(G)$ for an algebraic group G/k.

II. Root data and Existence and Isogeny Theorem

Goal: Classify split reductive groups using combinatorial data.

Definition Let $X, X^\vee$ be free $\mathbb{Z}$-modules with perfect pairing $\langle -,- \rangle : X \times X^\vee \to \mathbb{Z}$. If $\Phi \subseteq X$ is a finite set, given a bijection $\Phi \to \Phi^\vee$ ($\alpha \mapsto \alpha^\vee$), we say $(X, \Phi, X^\vee, \Phi^\vee)$ is a <u>root datum</u> if

i) $\langle \alpha, \alpha^\vee \rangle = 2$

ii) If $s_\alpha : X \to X$ is defined by ~~$s_\alpha(x) = x - \langle x, \alpha^\vee \rangle$~~ $s_\alpha(x) = x - \langle x, \alpha^\vee \rangle$, then $s_\alpha(\Phi) \subseteq \Phi$

iii) The group of automorphisms of X generated by $\{s_\alpha\}_{\alpha \in \Phi}$ is finite.

A root datum is <u>reduced</u> if $\mathbb{Q} \cdot \alpha \cap \Phi = \{\pm \alpha\}$.

~~Example~~ ~~(SL_2, T)~~ Want to associate a root datum to a split reductive group.

<u>Example</u> (SL_2, T)

$$X = X^*(T) = \underset{T \to \mathbb{G}_m}{\text{character lattice}} = \frac{\mathbb{Z}\chi_1 + \mathbb{Z}\chi_2}{\langle \chi_1 + \chi_2 \rangle} \quad \text{where } a\chi_1\left(\begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix}\right) = t_1^a$$
$$b\chi_2\left(\begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix}\right) = t_2^b.$$

$X^\vee = X_*(T) = \text{cocharacter lattice } \mathbb{G}_m \to T = \{a_1\lambda_1 + a_2\lambda_2 \mid a_i \in \mathbb{Z}, a_1 + a_2 = 0\}$.

Here
$$a_1\lambda_1 + a_2\lambda_2 : t \mapsto \begin{pmatrix} t^{a_1} & \\ & t^{a_2} \end{pmatrix}.$$

$\langle f, g \rangle =$ ~~[illegible]~~ $f \circ g : \mathbb{G}_m \to SL_2 \to \mathbb{G}_m =$ ~~[illegible]~~ $l \in X^* \mathbb{G}_m$.

For $\Phi, \Phi^\vee$ we use the Lie algebra.

$\mathrm{Ad}: SL_2 \to GL(\mathfrak{sl}_2)$

$g \mapsto$ (conjugation by g) (same definition for all groups).

Consider $\mathrm{Ad}|_T$. We have

$$\mathrm{Ad}\left(\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}\right) \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = (\chi_1 - \chi_2)\left(\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}\right) \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

The other basis vector gives $\chi_2 - \chi_1$.

$\Phi := \{\chi_1 - \chi_2, \chi_2 - \chi_1\}$ = the roots.

$\Phi^\vee = \{\lambda_1 - \lambda_2, \lambda_2 - \lambda_1\}$ (easy to see if we force them to pair to 2).

Check: $(\chi_1-\chi_2) \circ (\lambda_1 - \lambda_2): t \mapsto \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \mapsto t \cdot (t^{-1})^{-1} = t^2$; similarly for the other one.

The automorphism condition is easy to show and left as an exercise.

Note A different (and perhaps more standard) way of writing this root datum is $(\mathbb{Z}e, \pm 2e, \mathbb{Z}e^*, \pm e^*)$ where $e = \chi_1$, $e^* = \lambda_1 - \lambda_2$.

All four pieces of data in a root datum are important!

Example

$$\mathbb{G}_m^{r-1} \times SL_2, \quad \mathbb{G}_m^{r-1} \times PGL_2, \quad \mathbb{G}_m^{r-2} \times GL_2$$

Root data: $(\mathbb{Z}^r, \pm 2e_1, \mathbb{Z}, \pm e_1^*)$, $(\mathbb{Z}^r, \pm e_1, \mathbb{Z}^r, \pm 2e_1^*)$, $(\mathbb{Z}^r, \pm(e_1+e_2), \mathbb{Z}^r, \pm(e_1^* + e_2^*))$.

Langlands Dual Group (over $\mathbb{C}$).

G reductive group, the Langlands dual group $G^\vee$ is the reductive group corresponding to the dual root datum $(X, \Phi, X^\vee, \Phi^\vee) \longleftrightarrow (X^\vee, \Phi^\vee, X, \Phi)$.

Examples

G	$G^\vee$
SL_n	PGL_n
$SO(2n+1)$	$Sp(2n)$
$SO(2n)$	$SO(2n)$
GL_n	GL_n

The notion makes sense because of the following theorem:

Theorem (Existence and Isogeny Theorem)

Category of split Reductive groups $(G,T) \mapsto (X, \Phi, X^\vee, \Phi^\vee)$ Category of root data

ob: (G,T) $\longmapsto$ ob: reduced root data

morphisms: isogenies $(G,T) \to (G',T')$ modulo $T'/Z(G)$ morphisms: isogeny of root data.

is an equivalence of categories.