E1. Show that there exists an $\mathcal{N} \models \mathrm{PA}$ and an $a \in \mathcal{N} \smallsetminus \mathbb{N}$ so that a is definable in $\mathcal{N}$.

E2. Let $(A, <)$ be a dense total order without endpoints, and assume that A is homogeneous in the sense that (a, b) is isomorphic to A whenever $a, b \in A$ with $a < b$ (examples: $\mathbb{R}$, $\mathbb{Q}$). Let $\alpha(A)$ be the least ordinal which is not isomorphic to any subset of A. Prove that $\alpha(A)$ is a regular uncountable cardinal. Then give examples of such A, B with $|A| = |B| = \aleph_1$ and $\alpha(A) = \omega_1$ and $\alpha(B) = \omega_2$.

E3. Let T be a theory in the language consisting of a single binary relation symbol such that T has an infinite model which is an equivalence relation. Prove that T has two isomorphic countable models $\mathfrak{A}_0$ and $\mathfrak{A}_1$ such that $\mathfrak{A}_0$ is a *proper* elementary submodel of $\mathfrak{A}_1$.

E4. Let δ be any ordinal and let $\gamma = \omega^\delta$ (under ordinal exponentiation). Let $\mathcal{U} = \{S \subseteq \gamma\colon S$ has order type ω and is unbounded in $\gamma\}$. Prove that $|\mathcal{U}|$ is 0 or $|\gamma|^{\aleph_0}$.

E5. Let M model PA.

1. Show that there is no formula $\varphi(x, y)$ so that every subset $D \subseteq M$ definable (with parameters) is defined in M by $\varphi(x, b)$ for some b.
2. Show that for every $c \in M$, there is a formula $\varphi(x, y)$ so that every subset $D \subseteq [0, c)$ definable (with parameters) is defined in M by $\varphi(x, b)$ for some b.

E6. Prove or disprove: There exists a partial computable function f such that the domain of f and range of f are not computable but the graph of f is computable.

E7. Let $\kappa < \gamma$ be regular cardinals. Partially order $\kappa \times \gamma$ by saying $(\alpha_1, \beta_1) \leq (\alpha_2, \beta_2)$ iff $\alpha_1 \leq \alpha_2$ and $\beta_1 \leq \beta_2$. Then $x < y$ means $x \leq y$ and $x \neq y$. Prove that there is no ordinal α and function $f : \alpha \to \kappa \times \gamma$ such that f is both *cofinal* and *increasing*:

$$\forall x \in \kappa \times \gamma\ \exists \xi < \alpha\ \ [x < f(\xi)] \quad \text{and} \quad \forall \xi, \eta < \alpha\ \ [\xi < \eta \to f(\xi) < f(\eta)].$$

E8. If α, β are non-zero ordinals, then there is a largest ordinal which divides both of them. Here, δ divides α iff $\alpha = \delta\xi$ for some ξ.