

### Math 763. Homework 3

Due Thursday, October 3rd

In these problems (and everywhere else in the class), the ground field, which is denoted by  $k$ , is assumed to be algebraically closed.

Unless specified otherwise, we do not assume varieties to be either quasi-compact or separated (especially since the notion of separated-ness was not even discussed).

1. Denote by  $G = GL(n, k)$  the group of all invertible  $n \times n$  matrices with entries from  $k$ . Using the entries as coordinates, we embed  $GL(n, k)$  into the affine space  $\mathbb{A}^{n^2}$ . It is easy to see that  $G$  is open.

Show that  $G$  is an *algebraic group*: the multiplication map  $m : G \times G \rightarrow G$  and the inversion map  $i : G \rightarrow G$  are regular. (Note that  $G \times G$  is an open subset of  $\mathbb{A}^{2n^2}$ .)

2. Show that any quasi-compact variety is a noetherian topological space.

3. Let  $X$  be a variety. Show that  $X$  can be written as a locally finite union of irreducible components: there is a decomposition  $X = \bigcup X_\alpha$  where each  $X_\alpha$  is a closed irreducible subset, and each point of  $X$  has a neighborhood that meets only finitely many  $X_\alpha$ 's. The decomposition is unique if we assume that  $X_\alpha \not\subset X_\beta$  for  $\alpha \neq \beta$ . (This follows from the previous problem if  $X$  is quasi-compact, so the problem is only interesting if  $X$  fails to be quasi-compact.)

4. Let  $X$  and  $Y$  be varieties and  $f, g : X \rightarrow Y$  be regular maps. Prove that the subset

$$\{x \in X : f(x) = g(x)\} \subset X$$

is locally closed in  $X$ .

5. (The local ring of a point.) Let  $X$  be a variety; fix a point  $x \in X$ . Denote by  $O_x$  the *stalk* of the structure sheaf  $O_X$  at the point  $x$ . By definition, elements of  $O_x$  are equivalence classes of pairs  $(U, f)$ , where  $U$  is a neighborhood of  $x$  and  $f \in O_X(U)$  is a regular function; the pairs  $(U, f)$  and  $(U', f')$  are equivalent if  $x$  has a neighborhood  $V \subset U \cap U'$  such that  $f|_V = f'|_V$ .

Show that  $O_x$  is a local ring (with respect to the natural operations on functions) and that if  $X$  is affine,  $O_x$  is the localization of  $k[X]$  at the maximal ideal of  $x$ . ( $O_x$  is called the *local ring* of  $x$ ; its elements are *germs* of regular functions.)

6. Show that the local ring is the universal local invariant of a point: the local rings of points  $x \in X$  and  $y \in Y$  are isomorphic as  $k$ -algebras if and only if there exist two neighborhoods  $x \in U$  and  $y \in V$  and an isomorphism  $U \simeq V$  sending  $x$  to  $y$ .

7. Show that the local ring of a point  $x \in X$  is integral if and only if  $x$  lies on a unique irreducible component of  $X$ .