In these problems (and everywhere else in the class), the ground field, which is denoted by \( k \), is assumed to be algebraically closed.

1. Consider the curve \( X = V(x^3 + x^2 - y^2) \subset \mathbb{A}^2 \). Show that the function \( f = y/x \) is a birational isomorphism between \( X \) and \( \mathbb{A}^1 \). Find the domain of \( f \) and the domain of the map \( f^{-1} \). Find non-empty open subsets in \( X \) and \( \mathbb{A}^1 \) such that \( f \) provides an isomorphism of these sets.

2. Let \( X \subset \mathbb{A}^n \) be a hypersurface given by the equation \( f_{m-1}(x_1, \ldots, x_n) + f_m(x_1, \ldots, x_n) = 0 \), where \( f_{m-1} \) and \( f_m \) are non-zero homogeneous polynomials of degrees \( m - 1 \) and \( m \), respectively. Prove that if \( X \) is irreducible, it is rational. (Shafarevich, Problem I.3.5.)

3. Prove that an irreducible quadric (that is, given by a degree two equation) hypersurface is rational.

4. **Fiber product of varieties** Let \( X, Y, Z \) be three varieties, and let \( f : X \to Z \) and \( g : Y \to Z \) be regular maps. The fiber product (or Cartesian product) \( S := X \times_Z Y \) is a variety together with maps \( \phi : S \to X \) and \( \psi : S \to Y \) such that \( f \circ \phi = g \circ \psi \) and the triple \((S, \phi, \psi)\) is universal (for any other such triple \((S', \phi', \psi')\), there is a unique map \( S' \to S \) making the natural diagram commute).

    Show that the fiber product exists. Prove that it is always a locally closed subvariety of \( X \times Y \). What condition would imply that it is closed?

    (Remark: if \( X \hookrightarrow Z \) is an embedding of a locally closed subvariety, then the fiber product is the preimage \( g^{-1}(X) \).)

5. Let \( f : X \to Y \) be a morphism of varieties. For every point \( x \in X \), \( f \) induces a morphism of local rings \( f_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x} \).

    Prove that \( f \) is an isomorphism if and only if it is a homeomorphism and \( f_x \) is an isomorphism for all \( x \in X \).

6. Suppose \( X \) is separated. Prove that for any two affine open subset \( U, V \subset X \), the intersection \( U \cap V \) is affine. (Hint: consider \( U \times V \subset X^2 \).)

7. Let \( X \subset \mathbb{A}^n \) be an irreducible hypersurface. Consider the projection \( \pi : X \to \mathbb{A}^{n-1} \) onto one of the coordinate hyperplanes. Suppose that \( \pi \) is dominant.
Show that the induced map $k(\mathbb{A}^{n-1}) \hookrightarrow k(X)$ realizes $k(X)$ as a finite extension of $k(\mathbb{A}^{n-1})$ (this is almost proved in class). Put $\deg(\pi) := [k(X) : k(\mathbb{A}^{n-1})]$. Now show that there is a non-empty open subset $U \subset \mathbb{A}^n$ such that every $x \in U$ has exactly $\deg(\pi)$ preimages.