Math 763. Homework 9
Due Thursday, December 5th

1. Let $X \subset \mathbb{A}^n$ be the union of coordinate axes. Prove that X cannot be embedded in $\mathbb{A}^{n-1}$: there is no algebraic subset $Y \subset \mathbb{A}^{n-1}$ such that $X \simeq Y$.

Remark: It is also easy to construct a similar example where X is an *irreducible* curve.

2. Show that a singular cubic (irreducible) hypersurface in $\mathbb{P}^n$ is rational, provided it is not a cone.

3. Let $X = V(f) \subset \mathbb{P}^n$ be a hypersurface, where the homogeneous polynomial f is square-free. Show that the singular locus of X is given by equations

$$f = 0, \quad \frac{\partial f}{\partial x_i} = 0, \qquad (i = 0, \dots, n).$$

(If $\deg(f)$ is not divisible by the characteristic of k, the first equation follows from the rest.)

4. Let X and Y be two varieties. Show that $(x, y) \subset X \times Y$ is smooth if and only if $x \in X$ and $y \in Y$ are both smooth.

5. Prove that the property of a hypersurface to be smooth is open, in the following sense. Recall that degree d hypersurfaces in $\mathbb{P}^n$ are of the form $V(f)$, where f is a homogeneous polynomial in $n+1$ variables. If f is considered only up to scaling, it corresponds to a point in a projective space $\mathbb{P}^N$ (with homogeneous coordinates given by the coefficients of f). Denote by $S \subset \mathbb{P}^N$ the subset of points such that the corresponding f is square-free and $V(f)$ is smooth. Prove that S is open.

6. Keeping the notation of the previous problem, find the codimension of $\mathbb{P}^N - S$.

7. Let $X \subset \mathbb{P}^n$ be a projective variety. Denote by $Y \subset \mathbb{P}^n$ the *secant variety* of X: its is the union of lines $\ell \subset \mathbb{P}^n$ such that either ℓ meets X in at least two points (ℓ is secant) or ℓ meets X at a single point and is tangent to X at this point (ℓ is tangent). Show that Y is a projective variety.

Remark: Here is an equivalent description of Y, which you may use without a proof. Let $(x_0 : x_1)$ be the homogeneous coordinates on the line ℓ. For any homogeneous polynomial f on $\mathbb{P}^n$, its restriction $\mathbb{P}^n|_\ell$ is a homogeneous polynomial in x_0 and x_1. Then $\ell \in Y$ if and only if there exists a non-zero homogeneous polynomial g of degree 2 such that g divides $f|_\ell$ for all homogeneous polynomials f that vanish on X. (If g has two distinct zeros, ℓ is secant, if it has a double zero, ℓ is tangent.)

8. Suppose $X \subset \mathbb{P}^n$ is a smooth projective variety. Show that $\dim(Y) \leq 2\dim(X) + 1$, where Y is the secant variety from the previous problem.

Remark: This statement is important for the following reason. One can show that if Y is the secant variety of X, and $p \in \mathbb{P}^n - Y$, then the projection from p defines a closed embedding $X \hookrightarrow \mathbb{P}^{n-1}$. Iterating this construction, we can show that any smooth projective variety of dimension d can be embedded into the projective space of dimension $2d + 1$. The statement fails for singular varieties (this is similar to Problem 1).