Math 764. Homework 2
Due Friday, February 10th

Extension of a sheaf by zero.

Let X be a topological space, let $U \subset X$ be an open subset, and let $\mathcal{F}$ be a sheaf of abelian groups on U.

1. The extension by zero $j_!\mathcal{F}$ of $\mathcal{F}$ (here j is the embedding $U \hookrightarrow X$) is the sheaf on X that can be defined as the sheafification of the presheaf $\mathcal{G}$ such that

$$\mathcal{G}(V) = \begin{cases} \mathcal{F}(V), & V \subset U \\ 0, & V \not\subset U. \end{cases}$$

Is the sheafication necessary in this definition? (Or maybe $\mathcal{G}$ is a sheaf automatically?)

2. Describe the stalks of $j_!\mathcal{F}$ over all points of X and the espace étalé of $j_!\mathcal{F}$.

3. Verify that $j_!$ is the left adjoint of the restriction functor from X to U: that is, for any sheaf $\mathcal{G}$ on X, there exists a natural isomorphism

$$\mathrm{Hom}(\mathcal{F}, \mathcal{G}|_U) \simeq \mathrm{Hom}(j_!\mathcal{F}, \mathcal{G}).$$

(The *restriction* $\mathcal{G}|_U$ of a sheaf $\mathcal{G}$ from X to an open set U is defined by $\mathcal{G}|_U(V) = \mathcal{G}(V)$ for $V \subset U$.)

Side question (not part of the homework): What changes if we consider the version of extension by zero for sheaves of sets ('the extension by empty set')?

Examples of affine schemes.

4. Let R_α be a finite collection of rings. Put $R = \prod_\alpha R_\alpha$. Describe the topological space $\mathrm{Spec}(R)$ in terms of $\mathrm{Spec}(R_\alpha)$'s. What changes if the collection is infinite?

5. Recall that the image of a regular map of varieties is constructible (Chevalley's Theorem); that is, it is a union of locally closed sets. Give an example of a map of rings $R \to S$ such that the image of a map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is

(a) An infinite intersection of open sets, but not constructible.

(b) An infinite union of closed sets, but not constructible. (This part may be very hard.)

Contraction of a subvariety.

Let X be a variety (over an algebraically closed field k) and let $Y \subset X$ be a closed subvariety. Our goal is to construct a k-ringed space $Z = (Z, \mathcal{O}_Z) = X/Y$ that is in some sense the result of 'gluing' together the points of Y. While Z can be described by a universal property, we prefer an explicit construction:

- The topological space Z is the 'quotient-space' X/Y: as a set, $Z = (X - Y) \sqcup \{z\}$; a subset $U \subset Z$ is open if and only if $\pi^{-1}(U) \subset X$ is open. Here the natural projection $\pi : X \to Z$ is identity on $X - Y$ and sends all of Y to the 'center' $z \in Z$.
- The structure sheaf $\mathcal{O}_Z$ is defined as follows: for any open subset $U \subset Z$, $\mathcal{O}_Z(U)$ is the algebra of functions $g : U \to k$ such that the composition $g \circ \pi$ is a regular function $\pi^{-1}(U) \to k$ that is constant along Y. (The last condition is imposed only if $z \in U$, in which case $Y \subset \pi^{-1}(U)$.)

In each of the following examples, determine whether the quotient X/Y is an algebraic variety; if it is, describe it explicitly.

6. $X = \mathbb{P}^2$, $Y = \mathbb{P}^1$ (embedded as a line in X).

7. $X = \{(s_0, s_1; t_0 : t_1) \in \mathbb{A}^2 \times \mathbb{P}^1 : s_0 t_1 = s_1 t_0\}$, $Y = \{(s_0, s_1; t_0 : t_1) \in X : s_0 = s_1 = 0\}$.

8. $X = \mathbb{A}^2$, Y is a two-point set (if you want a more challenging version, let $Y \subset \mathbb{A}^2$ be any finite set).