Math 764. Homework 5
Due Friday, March 3rd

1. Fix a field k, and put $X = Spec\, k[x]$ and $Y = Spec\, k[y]$. Consider the morphism $f : X \to Y$ given by $y = x^2$. Describe the fiber product $X \times_Y X$ as explicitly as possible. (The answer may depend on k.)

2. (The Frobenius morphism.) Let X be a scheme of characteristic p: by definition, this means that $p = 0$ in the structure sheaf of X. Define the (absolute) Frobenius morphism $Fr_X : X \to X$ as follows: it is the identity map on the underlying set, and the pullback $Fr_X^*(f)$ equals f^p for any (local) function $f \in \mathcal{O}_X$.

Verify that this defines an affine morphism of schemes. Assuming X is a scheme locally of finite type over a perfect field, verify that Fr_X is a morphism of finite type (it is in fact finite, if you know what it means).

3. (The relative Frobenius morphism.) Let $X \to Y$ be a morphism of schemes of characteristic p. Put

$$\overline{X} := X \times_{Y, Fr_Y} Y,$$

where the notation means that Y is considered as a Y-scheme via the Frobenius map.

(a) Show that the Frobenius morphism Fr_X naturally factors as the composition $X \to \overline{X} \to X$, where the first map $X \to \overline{X}$ is naturally a morphism of schemes over Y (while the second map, generally speaking, is not). The map $X \to \overline{X}$ is called the *relative* Frobenius morphism.

(b) Suppose $Y = Spec(\overline{\mathbb{F}}_p)$, and X is an affine variety (that is, an affine reduced scheme of finite type) over $\overline{\mathbb{F}}_p$. Describe $\overline{X}$ and the relative Frobenius $X \to \overline{X}$ explicitly in coordinates.

4. Let X be a scheme over $\mathbb{F}_p$. In this case, the absolute Frobenius $Fr_X : X \to X$ is a morphism of schemes over $\mathbb{F}_p$ (and it coincides with the relative Frobenius of X over $\mathbb{F}_p$.

Consider the extension of scalars

$$X' = X_{\overline{\mathbb{F}}_p} = X \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p = X \times_{Spec(\mathbb{F}_p)} Spec(\overline{\mathbb{F}}_p).$$

Then Fr_X naturally extends to a morphism of $\overline{\mathbb{F}}_p$-schemes $X' \to X'$. Compare the map $X' \to X'$ with the relative Frobenius of X' over $\overline{\mathbb{F}}_p$.

5. A morphism of schemes is *surjective* if it is surjective as a morphism of sets. Show that surjectivity is preserved by base changes. That is, if $f : X \to Z$ is surjective and $g : Y \to Z$ is arbitrary, then $X \times_Z Y \to Y$ is surjective.

6. (Normalization) A scheme is *normal* if all of its local rings are integrally closed domains. Let X be an integral scheme. Show that there exists a normal integral scheme $\tilde{X}$ together with a morphism $\tilde{X} \to X$ that is universal in the following sense: any dominant morphism $Y \to X$ from a normal integral scheme to X factors through $\tilde{X}$. (Just like in the case of varieties, a morphism is *dominant* if its image is dense.)

7. Let X be a scheme of finite type over a field k. For every field extension $K \supset k$, put

$$X_K := X \otimes_k K = Spec(K) \times_{Spec(k)} X.$$

Show that X is geometrically irreducible (that is, the morphism $X \to Spec(k)$ has geometrically irreducible fibers) if and only if X_K is irreducible for all *finite* extensions $K \supset k$.