**Math 764. Homework 7**

Due Friday, March 31st

*Proper and separated morphisms.*

Each scheme $X$ has a maximal closed reduced subscheme $X^{red}$; the ideal sheaf of $X^{red}$ is the nilradical (the sheaf of all nilpotents in $\mathcal{O}_X$).

**1.** Let $f : X \to Y$ be a morphism of schemes of finite type. Consider the induced map $f^{red} : X^{red} \to Y^{red}$. Prove that $f$ is separated (resp. proper) if and only if $f^{red}$ is separated (resp. proper).

*Vector bundles.*

Fix an algebraically closed field $k$. Any vector bundle on $\mathbb{A}^1_k = Spec(k[t])$ is trivial, you can use this without proof. Let $X$ be the 'affine line with a doubled point' obtained by gluing two copies of $\mathbb{A}^1_k$ away from the origin.

**2.** Classify line bundles on $X$ up to isomorphism.

**3.** (Could be hard) Prove that any vector bundle on $X$ is a direct sum of several line bundles.

*Tangent bundle.*

**4.** Let $X$ be an irreducible affine variety, not necessarily smooth. Let $M$ be the $k[X]$-module of $k$-linear derivations $k[X] \to k[X]$. (These are globally defined vector fields on $X$, but keep in mind that $X$ may be singular.) Consider its generic rank $r := \dim_{k(X)} M \otimes_{k[X]} k(X)$. Show that $r = \dim(X)$.

**5.** Suppose now that $X$ is smooth. Show that the module $M$ is a locally free coherent module; the corresponding vector bundle is the *tangent bundle* $TX$.

**6.** Let $f : X \to Y$ be a morphism of algebraic varieties. Recall that a vector bundle $E$ over $Y$ gives a vector bundle $f^*E$ on $X$ whose total space is the fiber product $E \times_Y X$.

Suppose now that $X$ and $Y$ are affine and $Y$ is smooth. Let $E = TY$ be the tangent bundle to $Y$. Show that the space of $k$-linear derivations $k[Y] \to k[X]$ (where $f$ is used to equip $k[X]$ with the structure of a $k[Y]$-module) is identified with $\Gamma(X, f^*(TY))$.

**7.** Let $X$ be a smooth affine variety. Let $I_\Delta \subset k[X \times X]$ be the ideal sheaf of the diagonal $\Delta \subset X \times X$. Prove that there is a bijection

$$I_\Delta / I_\Delta^2 = \Gamma(X, \Omega^1_X),$$

where $\Omega^1_X$ is the sheaf of differential 1-forms (that is, the sheaf of sections of the cotangent bundle $T^\vee X$, which is the dual vector bundle of $TX$).