Math 764. Homework 1
Due Wednesday, February 12

1. Consider the smooth projective curve

$$C = V(x_1^2 + x_2^2 - x_0^2) \subset \mathbb{P}^2.$$

Determine the divisor of the function $f = (x_1/x_0) - 1$ on C. You can assume that the characteristic of k does not equal 2.

2. What is the divisor class group $\mathrm{Pic}(\mathbb{P}^n \times \mathbb{A}^m)$? $\mathrm{Pic}(\mathbb{P}^n \times \mathbb{P}^m)$? Justify.

3. Let X be a projective smooth irreducible variety. Prove that $\mathrm{Pic}(X) = 0$ if and only if X is a point.

4. Prove the statement made in class: if X is a smooth affine irreducible variety of dimension $\dim X > 0$, then for any divisor D, $\ell(D) = \infty$.

5. Let D be any divisor on $\mathbb{P}^n$. Recall that $\mathrm{Pic}(\mathbb{P}^n) = \mathbb{Z}$, with the isomorphism given by the degree map $\deg : \mathrm{Div}(\mathbb{P}^n) \to \mathbb{Z}$. Find a formula for $\ell(D)$ in terms of $\deg(D)$.

6. It is easy to see that for homogeneous polynomials

$$f, g \in k[x_0, x_1, x_2], \qquad \deg(f), \deg(g) > 0,$$

the set $V(f, g) \subseteq \mathbb{P}^2$ is not empty. (Using that $\mathbb{P}^2 - V(f)$ is affine.) Provide a different proof of this fact along the following lines:

Consider the ideal $I = (f, g) \subseteq k[x_0, x_1, x_2]$. Consider the space of homogeneous polynomials of a given degree m inside I. Give an upper bound on the dimension of this space and use it to show that $V(f, g)$ cannot be empty.

Remark: Using this approach, one can prove that $V(f, g)$ contains exactly $\deg(f)\deg(g)$ points, counting with appropriate multiplicity (provided f and g are have no common factors). This is known as *Bezout's Theorem* (in $\mathbb{P}^2$), and can be viewed as an algebraic calculation of the *intersection index* of the divisors (f) and (g). The statement extends to the zero locus of n homogeneous polynomials in $\mathbb{P}^n$.