Math 764. Homework 10

Due Wednesday, April 29th

1. Let X be a variety. By construction, a vector field τ on an open subset $U \subset X$ defines a derivation $D_\tau : \mathcal{O}_U \to \mathcal{O}_U$, which we can view as the directional derivative in the direction of τ. Prove that for any two vector fields τ_1, τ_2 on U, there is a vector field τ such that

$$D_\tau = D_{\tau_1} D_{\tau_2} - D_{\tau_2} D_{\tau_1}.$$

The vector field τ is called *the Lie bracket* of τ_1 and τ_2. (Remark: While $\mathcal{T}_X$ is a coherent sheaf, the Lie bracket does not turn it into a coherent Lie algebra, because the Lie bracket is not $\mathcal{O}_X$-linear. Rather, it becomes what is known as a *Lie algebroid* over $\mathcal{O}_X$.)

2. (Galois twist of a variety) Let $\sigma : k \to k$ be an automorphism of the ground field. Let X be a variety over k. Let us define on X a sheaf of k-algebras $\mathcal{O}_X^\sigma$ as follows: as a sheaf of rings, it coincides with $\mathcal{O}_X$, but the structure of k-vector spaces is twisted by σ^{-1} (i.e., multiplication by $a \in k$ in $\mathcal{O}_X^\sigma$ corresponds to multiplication by $\sigma^{-1}(a)$ in $\mathcal{O}_X$). Prove that there is a different structure of a variety over k on the topological space X such that $\mathcal{O}_X^\sigma$ is the structure sheaf of X with respect to this variety structure. Let us denote this variety by X^σ.

(Note: as originally posted, the problem used twist by σ in place of the twist by σ^{-1}. Either convention works in this problem, but for the next problem, it is important to use the correct version. Equivalently, the orginal wrong construction defined the structure of a k-vector space by restriction of scalars along $\sigma : k \to k$, while the correct operation uses the extension of scalars; in the case of an isomorphism, the two operations are mutually inverse.)

3. (The Frobenius twist) Suppose now that k has characteristic $p > 0$. Denote by $\phi : k \to k$ the Frobenius automorphisms $a \mapsto a^p$ (it is an automorphism because k is algebraically closed). Let X be a variety, and consider on X two sheaves of k-algebras: $\mathcal{O}_X$ and $\mathcal{O}_X^\phi$. Define a homomorphism $\Phi : \mathcal{O}_X^\phi \to \mathcal{O}_X$ by $\Phi(f) = f^p$. (Here we are using the Galois twist defined in the previous problem.) Show that homomorphism Φ yields a morphism of varieties $F : X \mapsto X^\phi$ (*The Frobenius morphism*) such that the corresponding map on underlying topological spaces is the identity.

(Here the word 'yields' means that Φ is identified with the natural map $\mathcal{O}_X^\phi = \mathcal{O}_{X^\phi} \to F_*\mathcal{O}_X = \mathcal{O}_X$.)

4. In the setting of the previous problem, suppose X is smooth of dimension n. Prove that $F_*\mathcal{O}_X$ is a locally free coherent sheaf on X^ϕ and find its rank. (Remark: more or less by construction, F is affine; in a fancier language, the problem asks you to show that F is finite and flat, and to find its degree.)

5. Let X be a smooth curve. Given a locally free coherent sheaf $\mathcal{F}$ on X, let us choose a point $x \in X$ and a subspace V in the fiber of $\mathcal{F}$ at x. Define $\mathcal{F}' \subset \mathcal{F}$ to be the subsheaf of sections s of $\mathcal{F}$ such that $s(x) \in V$. (Here $s(x)$ stands for the image of s in the fiber at x; of course, the condition is imposed only if s is defined at x.)

Prove that $\mathcal{F}'$ is also a locally free coherent sheaf on X. (This defines an operation on vector bundles on X: *modification* at x.)

6. In the setting of the previous problem, suppose $\mathcal{F}$ is locally free of rank r. Then $\Lambda^r \mathcal{F}$ is an invertible sheaf on X, and therefore it makes sense to talk about its degree. Put $\deg(\mathcal{F}) := \deg(\Lambda^r \mathcal{F})$. Find a formula for $\deg(\mathcal{F}')$, where $\mathcal{F}'$ is a modification of $\mathcal{F}$.

7. Let X be a separated variety. Let $\mathcal{I}_\Delta \subset \mathcal{O}_{X \times X}$ be the ideal sheaf of the diagonal $\Delta \subset X \times X$. Prove that there is a (canonical) isomorphism

$$\mathcal{I}_\Delta / \mathcal{I}_\Delta^2 \simeq \iota_* \Omega_\Delta.$$

Here $\iota : \Delta \hookrightarrow X \times X$ is the embedding of the diagonal. (This may be viewed as a generalization of the definition of the cotangent space at a point via its maximal ideal.)