Math 764. Homework 1
Due Wednesday, February 19

Elliptic curves. Let $p(x)$ be a square-free polynomial of degree 3, and suppose $\mathrm{char}(k) \neq 2$. Denote by X' the affine curve $y^2 = p(x)$ and by $X \subset \mathbb{P}^2$ its projective closure.

1. Show that X is smooth and irreducible.

2. There is exactly one "point at infinity" $p \in X - X'$. Provide a local parameter at this point.

3. Any regular function $\phi \in k[X']$ can be uniquely written in the form $f(x) + yg(x)$ for $f, g \in k[x]$. What is the order of pole of ϕ at p?

4. Find $\ell(N \cdot p)$ for divisor $N \cdot p \in \mathrm{Div}(X)$, $N \geq 0$. Conclude that X is not rational by comparing this to $\ell(N \cdot p')$ for a point $p' \in \mathbb{P}^1$).

5. Bertini's Theorem. Let $X \subset \mathbb{P}^n$ be a smooth subvariety. Prove that for almost all hyperplanes $H \subset \mathbb{P}^n$, the intersection $H \cap X$ is smooth and has multiplicity one. That is, the pullback of the divisor H to X is of the form $\sum Z_i$, where the prime divisors Z_i are smooth, they all appear with multiplicity one, and $Z_i \cap Z_j = \emptyset$ for $i \neq j$. Here 'almost all' means that in the projective space of all hyperplanes, there is a non-empty open subset of hyperplanes satisfying this condition.

(If you need a hint, look up Bertini's Theorem on Wikipedia. The 'usual' statement of Bertini's Theorem also includes the claim that if $\dim X > 1$, then $H \cap X$ is irreducible. There are many other 'Bertini-like' statements, some of which require that $\mathrm{char}(k) = 0$.)

Divisors on non-smooth varieties. Suppose that X is quasi-compact, irreducible, and *smooth in codimension one*: its singular locus has codimension two or more. We can still define $\mathrm{Div}(X)$ as before, and the divisor of a rational function makes sense, because the order of a function along a prime divisor can be computed away from the singular locus.

For example, let X be the quadratic cone $V(x^2 - yz) \subset \mathbb{A}^3$. ($\mathrm{char}(k) \neq 2$.) Its singular locus is the vertex $0 \in X$, which has codimension 2.

6. Show that the divisor of the function y is $2 \cdot Z$ for $Z = V(x, y)$.

7. Show that y is irreducible in the local ring $\mathcal{O}_{X,0}$.

8. Conclude that the divisor Z is not locally principal in the neighborhood of 0.