Math 764. Homework 3

Due Wednesday, February 26

1. Let $s : X \to E$ be a section of a vector bundle E over X. Show that the zero locus of s

$$\{x \in X : s(x) = 0 \in E_x\}$$

is a Zariski closed subset of X.

2. Let E and F be two vector bundles on X. Let $\phi : E \to F$ be a homomorphism. Suppose that it has *constant rank*: the rank of the linear operator $\phi_x : E_x \to F_x$ is independent of x. Show that in this case, the vector spaces $\ker(\phi_x)$ and $\operatorname{im}(\phi_x)$ $(x \in X)$ are fibers of naturally defined vector bundles on X, $\ker(\phi)$ and $\operatorname{im}(\phi)$. (The two vector bundles can also be defined categorically, by appropriate universal properties.)

3. A smooth variety is said to be parallelizable if its tangent bundle is trivial. Show that $\mathbb{P}^n$ is not parallelizable (unless $n = 0$).

4. Let G be an algebraic group: G is a variety with a group structure such that the inversion map and the multiplication map are regular. Prove that G is smooth and parallelizable. (Both facts follow from the simple transitive action of G on itself.)

5. Let X be an elliptic curve (i.e., the projective closure of the affine curve $y^2 = p(x)$ for square-free cubic polynomial $y^2 = p(x)$; here $\operatorname{char}(k) \neq 2$. Show that X is parallelizable. (This is immediate if one knows that X has a group structure, but proving it this way seems like an overkill.)

6. Show that the tangent bundle on $\mathbb{P}1$ is not trivial. (In combination with the previous problem, this gives another proof that an elliptic curve is not rational.)

7. Let X be a smooth affine variety. Consider the k-vector space of derivations $\partial : k[X] \to k[X]$ (i.e., k-linear maps such that $\partial(fg) = f\partial(g)+g\partial(f)$). Provide an isomorphism between the space of derivations and the space of vector fields on X (by definition, vector fields are sections of the tangent bundle). (Actually, the map is an isomorphism of $k[X]$-modules, not just k-vector spaces.)