Math 764. Homework 6
Due Wednesday, April 1st

In all these problems, we fix a topological space X; all sheaves and presheaves are sheaves on X.

Properties of morphisms of sheaves:

1. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves of sets. Show that a morphism $\phi : \mathcal{F} \to \mathcal{G}$ is an isomorphism if and only if ϕ induces isomorphisms of stalks $\mathcal{F}_x \to \mathcal{G}_x$ at all points $x \in X$.

2. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves of sets. Recall that a morphism $\phi : \mathcal{F} \to \mathcal{G}$ is a (categorical) monomorphism if and only if for any sheaf $\mathcal{F}'$ and any two morphisms $\psi_1, \psi_2 : \mathcal{F}' \to \mathcal{F}$, the equality $\phi \circ \psi_1 = \phi \circ \psi_2$ implies $\psi_1 = \psi_2$. Show that ϕ is a monomorphism if and only if it induces injective maps on all stalks.

3. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves of sets. Recall that a morphism $\phi : \mathcal{F} \to \mathcal{G}$ is a (categorical) epimorphism if and only if for any sheaf $\mathcal{G}'$ and any two morphisms $\psi_1, \psi_2 : \mathcal{G} \to \mathcal{G}'$, the equality $\psi_1 \circ \phi = \psi_2 \circ \phi$ implies $\psi_1 = \psi_2$. Show that ϕ is a epimorphism if and only if it induces surjective maps on all stalks.

4. Show that any morphism of sheaves can be written as a composition of an epimorphism and a monomorphism. (You should know what order of composition I mean here.)

Example:

5. Let X be the unit circle, and let $\mathcal{F}$ be the sheaf of C^∞-functions on X. Consider the morphism of sheaves of groups

$$\frac{d}{dt} : \mathcal{F} \to \mathcal{F}.$$

Here $t \in \mathbb{R}/2\pi\mathbb{Z}$ is the polar coordinate on the circle.

Is it injective? surjective? (i.e., a monomorphism? an epimorphism?) What is its kernel? its image?

Locally constant sheaves:

Definition. A sheaf $\mathcal{F}$ is *constant over an open set* $U \subset X$ if there is a subset $S \subset F(U)$ such that the map

$$\mathcal{F}(U) \to \mathcal{F}_x : s \mapsto s_x \text{ (the germ of } s \text{ at } x)$$

gives a bijection between S and $\mathcal{F}_x$ for all $x \in U$.

$\mathcal{F}$ is *locally constant* (on X) if every point of X has a neighborhood on which $\mathcal{F}$ is constant.

6. Recall that a *covering space* $\pi : Y \to X$ is a continuous map of topological spaces such that every $x \in X$ has a neighborhood $U \ni x$ whose preimage $\pi^{-1}(U) \subset U$ is homeomorphic to $U \times Z$ for some discrete topological space Z. (Z may depend on x; also, the homeomorphism is required to respect the projection to U.)

Show that if $\pi : Y \to X$ is a covering space, its sheaf of sections $\mathcal{F}$ is locally constant. Moreover, prove that this correspondence is an equivalence between the category of covering spaces and the category of locally constant sheaves. (If X is pathwise connected, both categories are equivalent to the category of sets with an action of the fundamental group of X.)