Matrix Groups and Applications

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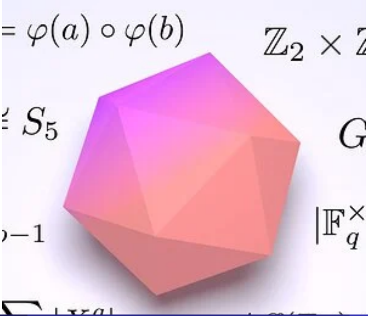
What is a Group?

A **group** is a mathematical structure consisting of a set of elements, along with an operation that combines any two elements to form a third element, satisfying certain rules. Groups are a central concept in abstract algebra and have applications in numerous fields such as geometry, number theory, and physics.

There are many different types and structures of groups; however, there are fundamental axioms all groups must follow.



Fundamental Axioms

We axiomatize a group as a structure $(G, \cdot, e)$, often simply written G, where

$x \cdot y$ is a two-variable function called product

x^{-1} is a one-variable function called inverse

e is a constant called unity/identity

Three Fundamental Axioms

The following are the three fundamental axioms which define a group G, where $x, y, z \in G$

1. $\forall x, y, z(x(yz) = (xy)z)$ - Associativity
2. $\forall x(xx^{-1} = e)$ - Inverse
3. $\forall(xe = x)$ - Identity

Propositions and Proofs

Utilizing the axioms mentioned in the previous page, we prove the following propositions:

1. $\forall x, y, z(zx = zy \implies x = y)$
2. $\forall x, y(xy = e \implies y = x^{-1}$
3. $\forall x, y(xy = e \implies yx = e)$

Abelian Groups

A group is called Abelian, or commutative, when it satisfies an extra property:

$$\forall x, y (xy = yx)$$

So they commute!

For an abelian group one often uses notation $(G, +, 0)$.

11.1 Group Homomorphisms

A ***homomorphism*** between groups $(G, \cdot)$ and $(H, \circ)$ is a map $\phi : G \to H$ such that

$$\phi(g_1 \cdot g_2) = \phi(g_1) \circ \phi(g_2)$$

for $g_1, g_2 \in G$. The range of ϕ in H is called the ***homomorphic image*** of ϕ.

Two groups are related in the strongest possible way if they are isomorphic; however, a weaker relationship may exist between two groups. For example, the symmetric group S_n and the group $\mathbb{Z}_2$ are related by the fact that S_n can be divided into even and odd permutations that exhibit a group structure like that $\mathbb{Z}_2$, as shown in the following multiplication table.

	even	odd
even	even	odd
odd	odd	even

We use homomorphisms to study relationships such as the one we have just described.

9.1 Definition and Examples

Two groups $(G, \cdot)$ and $(H, \circ)$ are ***isomorphic*** if there exists a one-to-one and onto map $\phi : G \to H$ such that the group operation is preserved; that is,

$$\phi(a \cdot b) = \phi(a) \circ \phi(b)$$

for all a and b in G. If G is isomorphic to H, we write $G \cong H$. The map ϕ is called an ***isomorphism***.

Matrix Groups

A matrix group is a group whose elements are matrices, and the group operation is matrix multiplication. These groups are subsets of the set of all $n \times n$ matrices that satisfy the axioms of a group under multiplication.

One of the most common matrix groups which I will touch on in this presentation is the General Linear Group($\mathrm{GL}_n(\mathbb{R})$)

General Linear Group

The General Linear Group over $\mathbb{R}$ is defined as:
$\mathrm{GL}_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det(A) \neq 0\}$
where:

1. $M_n(\mathbb{R})$ is the set of all $n \times n$ matrices with real entries.
2. $\det(A) \neq 0$ ensures that A is invertible.

Now, I will prove that the General Linear Group is indeed a group (closure, identity, inverse, associativity).

Another Example

Another example of an interesting matrix group is the rigid motions of a sphere, the group $SO(3)$

The simplest interesting matrix group, called SO(3), can be described in the following way: SO(3) $\cong$ to all positions of a globe on a fixed stand with the operation of rotations.

Three elements of SO(3) are pictured in the following image. Though the globe always occupies the same place in space, the three elements differ in the directions that various countries face.

SO(3) Image

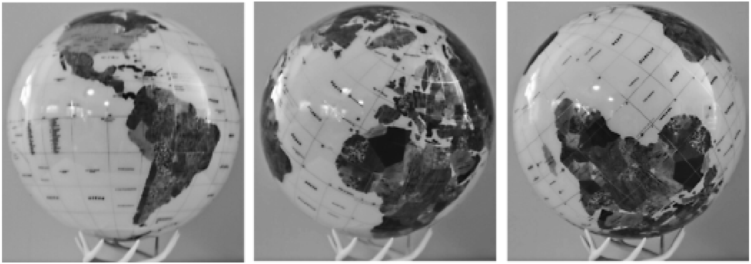


Figure 1. Three elements of $SO(3)$.

Neighborhood

A neighborhood is a set that contains the point p together with some open set that contains p. The concept of a neighborhood in mathematics is flexible and context-dependent. The radius (or size) of the neighborhood can be chosen based on the specific problem or constraints you are working with.

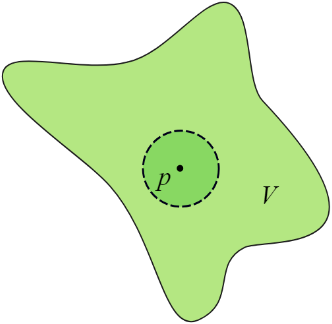


Figure: Neighborhood - Wikipedia

Homeomorphism

A homeomorphism is a bijective function between two topological spaces that is continuous in both directions and respects the structure of open neighborhoods, preserving their topological equivalence.

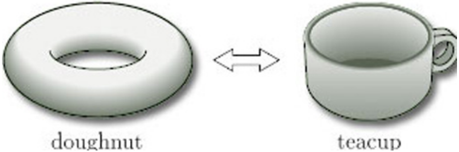


Figure: Homeomorphism - OpenLearn

Manifolds

A manifold is a topological space that locally resembles Euclidean space. Formally, a space M is a manifold if, for every point $p \in M$, there exists an open neighborhood of p that is homeomorphic to an open set in Euclidean space $\mathbb{R}^n$, where n is the dimension of the manifold.

This means that, near each point, the manifold looks like flat space (i.e., like $\mathbb{R}^n$) but can have a more complicated global structure.

Manifold Picture!

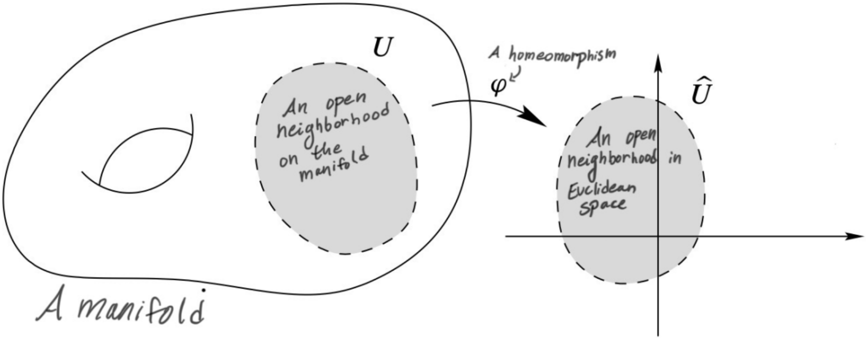


Fig. 1.2 A coordinate chart

Figure: Smooth Manifolds by John Lee, edited by Bella Finkel

Thurston's Geometrization Conjecture

Thurston's geometrization conjecture (now a theorem) states that each of certain three-dimensional topological spaces has a unique geometric structure that can be associated with it.

Euclidean Space

The Euclidean Space($\mathbb{E}^3$)corresponds to the "usual" 3D geometry we learn at school. It is the geometry of the flat real 3-dimensional vector space: $\mathbb{R}^3$

The Riemannian metric for $\mathbb{E}^3$ is the standard Euclidean metric, which in differential form is given by:
$ds^2 = dx^2 + dy^2 + dz^2$.

Here, dx, dy, dz represent the infinitesimal changes in the x, y, z-coordinates, respectively. This metric describes the infinitesimal distance between two nearby points in Euclidean space and reflects the flatness of $\mathbb{E}^3$

3-Sphere

The space The space $\mathbb{S}^3$ is the 3-dimensional analogue of the usual sphere in 2 dimensions. It is an isotropic space (all the directions play the same role).

The 3-sphere is the unit sphere of the 4-dimensional space $\mathbb{R}^4$. Thus a possible model X is:

$$X = \{(x, y, z, w) \in \mathbb{R}^4 \mid x^2 + y^2 + z^2 + w^2 = 1\}.$$

The Euclidean metric on $\mathbb{R}^4$ induces a Riemannian metric on X:

$$ds^2 = dx^2 + dy^2 + dz^2 + dw^2.$$

Thurstons Geometries Demo

Demo!

Citations

What is a Group?

Matrix Groups for Undergraduates

Introduction to Smooth Manifolds - John Lee

Manifold - Wolfram

How to see the eight Thurston Geometries - ENSAIOS MATEMATICOS

Thurston's Eight Model Geometries - Nachiketa Adhikari

Thank you!