ALGEBRAIC STACKS

KEVIN D.T. DAO

Abstract. These are notes for a talk on Algebraic Stacks for the algebraic stacks reading seminar. All mistakes here are my own and nothing here is original. Very sketchy notes in this version.

1. Definition of Algebraic Stack

Recall from Jeremy's talk that a **stack** is a fibred category satisfying descent. An algebraic stack will be an upgrade of what a stack is just like how algebraic spaces defined as an upgrade to fppf/étale sheaves.

Definition 1.1. A stack $\mathscr{X}/S$, where S is a scheme, is an **algebraic stack** if the following hold:

(1) the diagonal $\Delta_{\mathscr{X}} : \mathscr{X} \to \mathscr{X} \times_S \mathscr{X}$ is a representable morphism,
(2) and there exists a smooth surjective morphism $\pi : X \to \mathscr{X}$ from a scheme X.

Here, a **representable morphism** means being representable by algebraic spaces i.e. $f : \mathscr{X} \to \mathscr{Y}$ is representable if $U \to \mathscr{Y}$ is a scheme, then $\mathscr{X} \times_{\mathscr{Y}} U$ is an algebraic space.

Note that (1) ensures that (2) makes sense. In the literature, algebraic stacks might be referred to as **Artin stacks**.

2. Algebraic Stacks as a Category

Suppose you are given a algebraic S-stacks $a : \mathscr{X} \to \mathscr{Z}$ and $b : \mathscr{Y} \to \mathscr{Z}$. Then you can form the fibred category $\mathscr{Y} \times_{\mathscr{Z}} \mathscr{X}$.

Theorem 2.1. This fibred category is actually an algebraic stack over S.

Next, I want to talk about morphisms of algebraic stacks. To do this, we begin with properties of an algebraic stack.

Definition 2.2. Let P be a property that is stable in the smooth category. Then $\mathscr{X}/S$ has P if there exists a smooth surjective morphism $\pi : X \to \mathscr{X}$ such that X has P.

A morphism $f : \mathscr{X} \to \mathscr{Y}$ has P if there is a chart for f by schemes such that the morphism $h : X \to Y$ has property P

$$\begin{tikzcd}
X \arrow[r, "\text{smooth atlas}"] \arrow[rr, bend left, "h"] \arrow[dr] & \mathscr{X}' \arrow[r] \arrow[d] & Y \arrow[d, dashed, "\text{smooth atlas}"] \\
& \mathscr{X} \arrow[r, "f"] & \mathscr{Y}
\end{tikzcd}$$

It is part of the definition of a chart that X, Y are schemes.

The above definition can be used if f is not a representable (by algebraic spaces) morphism. On the other hand, if f is representable by algebraic spaces and P is a property stable w.r.t. the smooth topology on algebraic spaces, then one can define f as having P in the "standard" way.

In particular, since the diaonal of an algebraic stack is always representable it follows that the diagonal of a morphism of stacks is also representable. So we can make sense of separatedness conditions of algebraic stacks.

Definition 2.3. $f : \mathscr{X} \to \mathscr{Y}$ is **separated** if $\Delta_{\mathscr{X}/\mathscr{Y}}$ is proper.

$f : \mathscr{X} \to \mathscr{Y}$ is **quasiseparated** if it is quasicompact and quasiseparated.

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Why this definition? For schemes, the diagonal is always an immersion. In particular, a scheme is separated iff the diagonal is proper iff it is a closed immersion iff it is finite. So for the general setting of stacks, it makes sense to pick proper. Also, closed immersions are proper monomorphisms and since diagonals for schemes are already monomorphisms, it makes sense to only need to impose properness.

Example 2.4. Let $G := \mathbb{A}^1_k/\mathbb{Z}$. This is a group algebraic space. It is quasicompact but not quasiseparated. The diagonal of $B_k G$ is quasicompact but not quasiseparated.

Example 2.5 (Diagonal of a stack need not be an immersion)**.** Take BG and consider the diagonal. Let $\mathrm{Spec}(k) \to BG \times BG$ be given by $x \times x$ where $x : \mathrm{Spec}(k) \to BG$ is fixed. Then $G \cong \mathrm{Spec}(k) \times_{BG\times BG} BG$ but $G \to \mathrm{Spec}(k)$ is not an immersion.

This is different from what happens for algebraic spaces and schemes – their diagonals are immersions.

Now, let me list out some constructions one can still make with algebraic stacks.

- Relative Spec,
- Relative Proj,
- unions of open substacks,
- intersections of substacks, (??)**check this**

3. Examples of Algebraic Stacks

Example 3.1. Let X be an algebraic space, G/S a smooth group scheme, and assume G acts on X.

Define $[X/G]$ to be the stack as follows

(1) objects are triples $(T, \mathcal{P}, \pi)$ where T is an S-scheme, $\mathcal{P}$ is a G_T-torsor on the big étale site, and $\pi : \mathcal{P} \to X \times_S T$ is a G_T-equivariant morphism fo sheaves on (Sch/T).

(2) A morphism of triples is defined in the obvious way of pairs (f, f^b) where $f : T' \to T$ is a morphism, $f^b : \mathcal{P}' \to f^*\mathcal{P}$ is an isomorphism of $G_{T'}$-torsors, and the obvious diagram commutes.

(3) It is a stack because of descent for sheaves.

Now we check the axioms of an algebraic stack.

(1) First, check the diagonal is representable.

This is equivalent to showing that the Isom-presheaves are algebraic spaces. But one can work étale locally to do this. **Omitted.**

(2) Secondly, I claim that $q : X \to [X/G]$ is a smooth covering. This map sends a morphism $T \to X$ to the triple of $(T \to S, \mathcal{P} = X \times_S G_T, \rho_T : \mathcal{P} \to X \times_S T)$ i.e. $T \to X$ goes to the trivial G_T-torsor and ρ_T is just the action map.

The fibre product $X \times_{[X/G]} T$ for an S-scheme T is precisely $\mathcal{P}$, the G_T-torsor determined by $T \to [X/G]$. The map $\mathcal{P} \to T$ is the projection map, so it is clearly smooth and surjective.

Example 3.2. Let $\mathbb{G}_m$ act trivially on $\{*\}$. Then $[\{*\}/\mathbb{G}_m]$ has for its objects $[\{*\}/\mathbb{G}_m](T)$ the groupoids of diagrams
$$\begin{tikzcd} \mathcal{P} \arrow[r, "s"] \arrow[d] & \{*\} \\ T & \end{tikzcd}$$
Here, $\mathcal{P}$ is a $\mathbb{G}_m$-torsor on T so $\mathcal{P}$ is a a line bundle on the scheme T. The $\mathbb{G}_m$-equivairant morphism π corresponds then to a choice of section of this line bundle. In general, one writes $B_S G := [\{*\}/G]$ for the classifying stack of G/S, where G is a smooth group scheme over S.

Here is a list of properties of classifying stacks.

- $B_S(G \times H) \cong B_S(G) \times B_S(H)$ for smootha affine group schemes,
- $[X \times_S Y/G \times_S H] \cong [X/G] \times_S [Y/H]$,
- so $[\mathbb{A}^n/\mathbb{G}_m^n] \cong [\mathbb{A}^1/\mathbb{G}_m]^{\times n}$,
- if $H \subseteq G$, one can use induction to define a morphism $BH \to BG$, and in this case, $BH \times_{BG} S \cong [G/H]$,
- $B\,\mathrm{GL}_n$ is an algebraic stack, but is not a DM stack (defined later).

Example 3.3 ($Vec_{X/S}$)**.** Let $f : X \to S$ be a proper flat morphism of algebraic spaces which étale locally on S is projective.

Let $Vec_{X/S}$ be the fibred category whose fibre over $T \to S$ is the groupoid of locally free sheaves of finite rank on $X_T := X \times_S T$. Then $Vec_{X/S}$ is an algebraic stack. **See Olsson for a sketch of the proof.**

4. Definition of a Deligne-Mumford Stack

Definition 4.1. An algebraic stack $\mathscr{X}/S$ is a **Deligne-Mumford stack** (DM stack) if there exists a scheme X together with an étale surjective morphism $\pi : X \to \mathscr{X}$.

Heuristically, one is supposed to think about DM stacks as having no "infinitesimal automorphisms".

Here's the precise formulation. First, assume $\Delta_{\mathscr{X}}$ is of finite presentation (a common assumption). Then $\Delta_{\mathscr{X}}$ is formally unramified iff Aut_x is a reduced finite k-group for all algebraically closed fields and $x \in \mathscr{X}(k)$. For ($\Longrightarrow$), one uses the fact

$$\mathrm{Aut}_x \cong \mathrm{Spec}(k) \times_{\mathscr{X} \times_S \mathscr{X}} \mathscr{X}$$

and the fact a finite type formally unramified scheme over k is a finite disjoint union of $\mathrm{Spec}(k)$. For ($\Longleftarrow$), one can appeal to the fact that $P := U \times_{\mathscr{X} \times_S \mathscr{X}} \mathscr{X} \to U$ is formally unramified iff for all $z \in U$ the map $P_z \to \mathrm{Spec}(k(z))$ is formally unramified. By descent, one can pass to showing $P_\Omega \to \mathrm{Spec}(\Omega)$ is formally unramified for $k(z) \hookrightarrow \Omega$ the algebraic closure. But P_Ω is either empty or the automorphism group of $x \in X(\Omega)$.

Theorem 4.2. An algebraic stack $\mathscr{X}/S$ is a DM stack if and only if the diagonal is **formally unramified**. A morphism of schemes $Z \to W$ is formally unramified if for nilpotent closed embedding of affine schemes $S_0 \hookrightarrow S$, we have an injective morphism $Z(S) \hookrightarrow Z(S_0) \times_{W(S_0)} \times W(S)$. This is the same as saying $\Omega^1_{Z/W} = 0$.

How to interpret this as saying the automorphism groups are discrete and reduced? And equivalently finite and reduced if Δ is qc? **Sketch it out in the talk.**

5. $\mathscr{M}_g$ is a Deligne-Mumford Stack if $g \geq 2$

Here are the steps that one takes to prove the claim in the section title.

(1) Show that $\mathscr{M}_g$ is a stack.
(2) Show that $\mathscr{M}_g \cong [\widetilde{M}_g/G]$ for some algebraic space (quasiprojective variety) $\widetilde{M}_g$ with an action by $G = \mathrm{GL}_{5g-5}$. This shows $\mathscr{M}_g$ is an algebraic stack.
(3) Check that the automorphism groups are discrete and reduced to deduce that $\mathscr{M}_g$ is a DM stack.

The details can be found in Olsson's book 8.4.3. Here, we roughly sketch out the key parts of the proof.

(1) $\mathscr{M}_g$ is a stack due to effectiveness of descent for schemes.
(2) This is the harder step. First, define $\widetilde{M}_g$ as the functor that takes a scheme T to isomorphism classes of pairs $(f : C \to T, \sigma : O_T^{5g-5} \xrightarrow{\sim} f_* L_{C/T})$. One can embedd $C \hookrightarrow \mathbb{P}_T^{5g-6}$ using σ and the bundle $f_* L_{C/T}$. In any case, one can then realize $\widetilde{M}_g$ as a quasiprojective subscheme of some Hilbert scheme. The action of GL_{5g-5} comes from the obvious action on $\widetilde{M}_g$. Checking that $\mathscr{M}_g \cong [\widetilde{M}_g/G]$ boils down to showing that the natural map $\pi : \widetilde{M}_g \to \mathscr{M}_g$ induces the desired isomorphism (and one checks this one "S-points").
(3) To check that $\mathscr{M}_g$ is a DM stack, one studies the automorphism groups. This step is not as bad as it sounds.

    Let $A' \to A$ be a surjective morphism with squarezero kernel. Then there is a map $\mathrm{Aut}_k(C)(A') \to \mathrm{Aut}_k(C)(A)$ which we wish to show is injective. Here, discreteness is a consequence of the fact that the automorphism groups of genus $g \geq 2$ curves are finite. See Olsson p. 187 for the proof.