Math 763.
The Nullstellensatz.

The goal of this note is to prove Hilbert's Nullstellensatz (which translates as Zero Locus Theorem). There are many different proofs, my goal here is to give a short relatively self-contained proof (without relying on, for instance, Noether's Normalization).

1. The statement(s)

The following statements can all be considered versions of the Nullstellensatz (The references are to Milne's notes):

Theorem 1 (The Nullstellensatz, Theorem 2.11)**.** *Suppose k is algebraically closed. For any ideal $J \subset k[x_1, \dots, x_n]$,*

$$I(V(J)) = \sqrt{J}.$$

(Note that the inclusion $I(V(J)) \supset \sqrt{J}$ is obvious.)

Theorem 2 ('Weak Nullstellensatz', Theorem 2.6)**.** *Suppose k is algebraically closed. If $J \subset k[x_1, \dots, x_n]$ has no common solutions (that is, $V(J) = \emptyset$), then $J = (1)$.*

Proposition 3. *Any maximal ideal in $k[x_1, \dots, x_n]$ is of the form*

$$\begin{aligned}\mathfrak{m}_a &= (x - a_1, \dots, x - a_n) \\ &= \ker(ev_a : k[x_1, \dots, x_n] \to k : f(x_1, \dots, x_n) \mapsto f(a_1, \dots, a_n))\end{aligned}$$

for some $a = (a_1, \dots, a_n) \in k^n$.

Lemma 4 (Algebraic Nullstellensatz=Zariski's Lemma, Lemma 2.7)**.** *Let $L \supset k$ be a field extension. Suppose that L is finitely generated as a k algebra. Then L is a finite extension.*

2. The proof of the Algebraic Nullstellensatz

Suppose

$$L = k(x_1, \dots, x_d)$$

is a transcendental extension of k that is finitely generated as a k-algebra. Let us arrive at a contradiction.

Example. For instance: the field of rational functions $k(z)$ is a transcendental extension of k. We claim that it is not a f.g. k-algebra. This is reasonably clear: no finite collection of rational functions $P_i(z)/Q_i(z)$ can generate $k(z)$. Indeed, consider the algebra $A \subset k(z)$ generated by $P_i(z)/Q_i(z)$. The denominator of any $g \in A$ is of the form $\prod Q_i(z)^N$, so only finitely many irreducible polynomials are allowed in the denominator. However, there are infinitely many irreducible polynomials in $k[z]$ (is it clear why?), so $A \subsetneq k(z)$.

Without losing generality, we may assume that L is finite over $k(x_1)$, otherwise, replace k with $k(x_1)$ and keep going.

Since L is finite over $k(x_1)$ but transcendental over k, x_1 must be transcendental over k. We therefore have to prove a generalization of the above example:

Let L be a finite extension of the field of rational functions $k(z)$ (where $z = x_1$). Then L is not a f.g. k-algebra.

Let $e_1, \dots, e_d$ be a $k(z)$-basis of L. We can completely describe L by structure constants ('multiplication table')

$$e_i \cdot e_j = \sum c_{ij}^k(z) e_k \qquad c_{ij}^k(z) \in k(z).$$

If $x_1, \dots, x_n$ is a set of elements of L, we can write them in the basis as

$$x_i = \sum f_i^j(z) e_j \qquad f_i^j(z) \in k(z).$$

Now it is clear that if A is the k-algebra generated by $x_i's$, then for any element

$$g = \sum g_i(z) e_i \in A \qquad g_i(z) \in k(z),$$

the denominators of g_i's are products of denominators of c_{ij}^k's and f_i^j's. Thus, only finitely many irreducible polynomials may appear in the denominators, and we see that $A \subsetneq L$, as claimed. □

3. More proofs

Let us now derive the remaining statements.

Proof of Proposition 3. Clearly, ideals $\mathfrak{m}_a$ are maximal (being the kernels of surjective maps onto k). Conversely, let $\mathfrak{m}$ be any maximal ideal. By Lemma 4, $k[x_1, \dots, x_n]/\mathfrak{m}$ is identified with k as a k-algebra. Hence, $\mathfrak{m}$ is the kernel of a homomorphism of k-algebras

$$k[x_1, \dots, x_n] \mapsto k[x_1, \dots, x_n]/\mathfrak{m} = k.$$

However, any such homomorphism is an evaluation map. □

Proof of Theorem 2. In fact, the theorem is equivalent to Proposition 3. Indeed,

$$V(J) = \{a \in k^n : \mathfrak{m}_a \supset J\},$$

so $V(J) = \emptyset$ iff J is not contained in any ideal $\mathfrak{m}_a$. On the other hand, $J = (1)$ iff J is not contained in any maximal ideal. □

Proof of Theorem 1. Clearly, the Weak Nullstellensatz is a special case of the Nullstellensatz. However, it also implies the Nullstellensatz via the so-called Rabinowitsch Trick.

Suppose $F(x_1, \dots, x_n) \in k[x_1, \dots, x_n]$ satisfies $f|_{V(J)} = 0$, and let us show that $F^k = 0$ for some k. Add an extra variable x_0 and consider the system of equations in $x_0, \dots, x_n$ of the form

$$\begin{cases} f(x_1, \dots, x_n) = 0, & f \in J \\ x_0 F(x_1, \dots, x_n) - 1 = 0. \end{cases}$$

(It is enough to let f run over a set of generators of J only; by Hilbert's Basis Theorem, this means that we are basically looking at a finite system of equations... but this is irrelevant here.) This system is incompatible: the first set of equations says that $(x_1, \dots, x_n) \in V(J)$, and hence $F(x_1, \dots, x_n) = 0$, while the last equation says that $x_0 = 1/F(x_1, \dots, x_n)$. Hence the Weak Nullstellensatz applies.

By the Weak Nullstellensatz, there is a linear combination

$$1 = \sum_i g_i(x_0, \dots, x_n) f_i(x_1, \dots, x_n) + g(x_0, \dots, x_n)(x_0 F(x_1, \dots, x_n) - 1)$$

for some $f_i \in J$ and $g, g_i \in k[x_0, \dots, x_n]$. Plug in $x_0 = 1/F$ in the field $k(x_1, \dots, x_n)$. We then get an identity in $k(x_1, \dots, x_n)$ of the form

$$1 = \sum_i g_i(1/F, x_1, \dots, x_n) f_i(x_1, \dots, x_n).$$

Now clear the denominators. $\square$

4. Remarks

It is instructive to think about the Rabinowitsch trick purely algebraically. The idea is to consider the expansion of the ideal $J \subset k[x_1, \dots, x_n]$ to an ideal in $k[x_1, \dots, x_n, F^{-1}]$. The condition that $F|_{V(J)} = 0$ implies that this expansion is the improper ideal (why?), which is equivalent to $F \in \sqrt{J}$.

Note also that for tautological reasons,

$$I(V(J)) = \bigcap_{a : \mathfrak{m}_a \supset J} \mathfrak{m}_a,$$

which thanks to Proposition 3 is the same as

$$I(V(J)) = \bigcap_{\mathfrak{m} \text{ is maximal}, \mathfrak{m} \supset J} \mathfrak{m}.$$

The intersection

$$J' := \bigcap_{\mathfrak{m} \text{ is maximal}, \mathfrak{m} \supset J} \mathfrak{m}$$

is basically the Jacobson radical of J. 'Basically' here refers to a terminological issue: usually, we talk about the Jacobson radical of a ring, but not of an ideal. It is more precise to say that J' is the preimage of Jacobson's radical of $k[x_1, \dots, x_n]/J$ under the projection

$$k[x_1, \dots, x_n] \to k[x_1, \dots, x_n]/J. \tag{5}$$

On the other hand,

$$\sqrt{J} = \bigcap_{\mathfrak{p} \text{ is prime}, \mathfrak{p} \supset J} \mathfrak{p}$$

is the preimage of the nilradical under (5). Thus, modulo Proposition 3, the Nullstellensatz is the claim that the two radicals of $k[x_1, \ldots, x_n]/J$ coincide. Since any finitely-generated k algebra is of this form, we arrive at the following claim.

Proposition 6. *For any finitely generated k-algebra, the nilradical and Jacobson's radical coincide.* □

I am tempted to call Proposition 6 'Rabinowitsch's Theorem', because this is essentially the algebraic version of what Rabinowitsch's Trick proves.

Exercise 7. Show that Proposition 6 holds even if k is not algebraically closed. (The same argument works.)

(Commutative) rings A whose nilradical is equal to Jacobson's radical are called *Jacobson rings* (a.k.a *Hilbert rings*). There is a generalized way of looking at the Nullstellensatz as follows:

Theorem 8. *Suppose A is a commutative Jacobson ring. Then any finitely generated A-algebra B is a Jacobson ring as well. Moreover, for any maximal ideal $J \subset B$, the pullback of J to A is a maximal ideal $I \subset A$, and B/J is a finite extension of A/I.*

However, the proof of Theorem 8 is a bit more technical.