UW–Madison Putnam Club

October 1, 2025 — Analysis

1. [*Putnam and Beyond*] Is there a function $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$(f \circ f \circ f)(x) = x^3 \quad \text{and} \quad (f \circ f \circ f \circ f)(x) = x^5 \quad \text{for all } x \in \mathbb{R}?$$

2. [Putnam 2011 A2] Find all differentiable functions $f\colon \mathbb{R} \to \mathbb{R}$ such that for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$f'(x) = \frac{f(x+n) - f(x)}{n}.$$

3. [*Putnam and Beyond*] Let $a \in (0,1)$ and suppose $f\colon \mathbb{R} \to \mathbb{R}$ satisfies

$$\lim_{x\to 0} f(x) = 0 \quad \text{and} \quad \lim_{x \to 0} \frac{f(x) - f(ax)}{x} = 0.$$

Show that $\lim_{x\to 0} \frac{f(x)}{x} = 0$.

4. [Putnam 2006 B2]] Let $X = \{x_1, \ldots, x_n\}$ be a set of real numbers. Show that there exists a nonempty subset $S \subset X$ and $m \in \mathbb{Z}$ such that

$$\left| m + \sum_{s \in S} s \right| \le \frac{1}{n+1}.$$

5. Let $f\colon [-1,1] \to \mathbb{R}$ be infinitely differentiable, and suppose $f^{(n)}(x) \ge 0$ for all $x \in (-1,1)$ and $n \in \mathbb{N}$. Show that the Taylor series of f centered at zero converges to f on $(-1,1)$.

6. [Putnam 2017 A3] Let f and g be positive continuous functions on $[0,1]$. Suppose $\int_0^1 f = \int_0^1 g$ but $f \neq g$. Given $n \in \mathbb{Z}$, define

$$I_n \coloneqq \int_0^1 \frac{f(x)^{n+1}}{g(x)^n}\, \mathrm{d}x.$$

Show that the sequence $I_{-1}, I_0, I_1, \ldots$ is increasing and $\lim_{n\to\infty} I_n = \infty$.

7. [*Putnam and Beyond*] Does there exist a continuous function $f\colon [0,1] \to \mathbb{R}$ that assumes every element of its range an even (and finite) number of times?

Hints

5. If $0 \leq t \leq x \leq 1$, then $\frac{x-t}{1-t} \leq x$. Try this in the Taylor remainder.

6. Cauchy–Schwarz