UW–Madison Putnam Club

October 16, 2024 — Probability

1. Shanille O'Keal shoots free throws on a basketball court. She hits the first and misses the second, and thereafter the probability that she hits the next shot is equal to the proportion of shots she has hit so far. What is the probability that she hits exactly 50 of her first 100 shots? [2002 Putnam B-1]

2. Three real numbers are chosen randomly and uniformly from the interval $[0, 1]$. What is the probability that they are the three side lengths of some triangle?

3. You have coins $C_1, C_2, \ldots, C_n$. For each k, coin C_k is biased so that, when tossed, it has probability $\frac{1}{2k+1}$ of falling heads. If the n coins are tossed, what is the probability that the number of heads is odd? Express the answer as a rational function of n. [2001 Putnam A-1]

4. How many subsets of $1, 2, \ldots, n$ contain no two consecutive elements?

5. Consider the following game played with a deck of $2n$ cards numbered from 1 to $2n$. The deck is randomly shuffled and n cards are dealt to each of two players A and B. Beginning with A, the players take turns discarding one of their remaining cards and announcing the number. The game ends as soon as the sum of the numbers on the discarded cards is divisible by $2n + 1$. The last person to discard wins the game. If we assume optimal strategy by both A and B, what is the probability that A wins? [1993 Putnam B-2]

6. Real numbers are chosen uniformly and independently at random from the interval $[0, 1]$. Let n denote the random integer at which the sum of the first n numbers first exceeds 1. Show that $\mathbb{E}n = \mathrm{e}$. [MIT Putnam seminar]

7. Suppose X is a random variable that takes on only nonnegative integer values, with $\mathbb{E}X = 1$, $\mathbb{E}X^2 = 2$, and $\mathbb{E}X^3 = 5$. Determine the smallest possible value of the probability of the event $X = 0$. [2014 A-4]

8. We pick n points uniformly and independently at random on a circle. What is the probability that the center of the circle will lie in the convex hull of those points?

9. Let k be a positive integer. Suppose that the integers $1, 2, \ldots, 3k + 1$ are written down in random order. What is the probability that at no time during the process, the sum of the integers that have been written up to that time is a positive integer divisible by 3? Your answer should be in closed form, but may include factorials. [2007 Putnam A-3]

Hints

1. Prove that if she attempts n shots, the probability she hits any number of shots from 1 to $n - 1$ is equal.

3. Consider the expression $(2/3 - 1/3)(4/5 - 1/5)(6/7 - 1/) \cdots (2n/(2n + 1) - 1/(2n + 1))$ and think what it means probabilistically.

4. Find a bijection between the k-element subsets of $1, 2, \ldots, n$ with no consecutive elements and all k-element subsets of $1, 2, \ldots, n - k + 1$.

5. The stuff about random shuffling and probabilities is a red herring: B always wins.

8. Find the probability that the hull does *not* contain the center of the circle.