UW–Madison Putnam Club

October 23, 2024 — Probability & Inequalities

1. Consider n balls independently and uniformly distributed in m boxes. What is the probability that exactly k boxes remain empty? [*Putnam and Beyond*]

2. Two real numbers x and y are chosen independently and uniformly in the interval $(0, 1)$. What is the probability that the closest integer to x/y is even? Express the answer in the form $r + \pi s$, where r and s are rational numbers. [Putnam 1993 B3]

3. Let π be a uniformly random permutation of the first n positive integers. (So π is drawn uniformly from the symmetric group S_n.) What is the probability that the numbers 1 and 2 lie within the same cycle in π? [*Putnam and Beyond*]

4. Let A be a $2n \times 2n$ random matrix with independent entries equal to 0 or 1 with equal probability. Find the expected value of $\det(A - A^\top)$. [Putnam 2016 B4]

5. Let $f\colon [0,1]^2 \to \mathbb{R}$ be continuous. Show that

$$\int_0^1 \left(\int_0^1 f(x, y)\, \mathrm{d}x \right)^2 \mathrm{d}y + \int_0^1 \left(\int_0^1 f(x, y)\, \mathrm{d}y \right)^2 \mathrm{d}x \le \left(\int_0^1 \int_0^1 f(x, y)\, \mathrm{d}x\, \mathrm{d}y \right)^2 + \int_0^1 \int_0^1 f(x, y)^2\, \mathrm{d}x\, \mathrm{d}y.$$

   [Putnam 2004 A6]

6. Given $a_1, \ldots, a_n \ge 0$, show that

$$(1 + a_1)(1 + a_2) \cdots (1 + a_n) \ge (1 + \sqrt[n]{a_1 a_2 \cdots a_n})^n.$$

   [Essentially Putnam 2003 A2]

7. Let m and n be positive integers. Show that

$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!}{m^m} \frac{n!}{n^n}.$$

   [Putnam 2004 B2]

8. Find the maximal value of the ratio

$$\frac{\left(\int_0^1 f(x)\, \mathrm{d}x \right)^3}{\int_0^1 f(x)^3\, \mathrm{d}x}$$

   as f ranges over all continuous functions $f\colon [0,1] \to (0, \infty)$.

Hints

3. Consider the complementary probability.
4. Represent the determinant via permutations and study the cycle type.
5. The difference can be expressed as an integral of a square.
6. What inequality does the geometric mean suggest?
7. There are many possible solutions. The binomial theorem points to one.