Putnam Club
Polynomials

Brian Lawrence

Oct. 2022

1 Resources and other competitions

- Art of Problem Solving forums https://artofproblemsolving.com/community/c5_contests_amp_programs
- International Math Olympiad (IMO) https://www.imo-official.org/
- IMO Shortlist
- National olympiads (USAMO, etc.)

2 Problems on polynomials

1. (interpolation)

   Let $x_0, \ldots, x_n$ be distinct real numbers, and let $y_0, \ldots, y_n \in \mathbb{R}$. Show that there is a unique polynomial p of degree at most n such that $p(x_i) = y_i$ for $i = 0, \ldots, n$.

2. (Vandermonde)

   Let $a_0, \ldots, a_n \in \mathbb{R}$. Let M be the $(n+1) \times (n+1)$ matrix whose (i, j) entry is a_i^j. Show that

$$\det M = \prod_{p<q} (a_q - a_p).$$

3. (Chebyshev)

   (a) Let n be a positive integer. Show that there exists a polynomial $T_n(x)$ of degree n such that $\cos n\theta = T_n(\cos\theta)$ for all $\theta \in \mathbb{R}$.

   (b) Say that a polynomial $p\colon \mathbb{R} \to \mathbb{R}$ *has self-control* if, for all $x \in [-1, 1]$, we have $p(x) \in [-1, 1]$. Show that, among all polynomials of degree n having self-control, T_n has the largest leading coefficient.

4. Let p be a polynomial with integer coefficients, and let a and b be arbitrary integers. Show that $p(a) - p(b)$ is divisible by $a - b$.

5. (USAMO 1977) If a and b are two of the roots of $x^4 + x^3 - 1 = 0$, prove that ab is a root of $x^6 + x^4 + x^3 - x^2 - 1 = 0$.

6. (Russia 2022) What is the smallest positive integer a such that there exist integers b and c, such that the polynomial $ax^2 + bx + c$ has two distinct real roots, and both those roots lie in the interval $[0, 1/1000]$?

7. (India 2021-22) Suppose that P is the polynomial of least degree with integer coefficients such that
$$P\left(\sqrt{7}+\sqrt{5}\right) = 2\left(\sqrt{7}-\sqrt{5}\right)$$
Find $P(2)$.

8. (Iran 2022) Fix an integer $n > 2$. Amin and Ali take turns playing the following game. At each step, the player chooses some index $i \in \{0, 1, \ldots, n\}$ that has not yet been chosen; the player also chooses a rational number $a_i \in \mathbb{Q}$. Ali takes the first turn. The game ends when all indices $0, 1, \ldots, n$ have been chosen. Ali wins if the polynomial equation
$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$
has a rational root; otherwise, Amin wins.

   Find all n for which Ali can win, no matter how Amin plays.

9. (Putnam 2017 A2) Let $Q_0(x) = 1$, $Q_1(x) = x$, and
$$Q_n(x) = \frac{(Q_{n-1}(x))^2 - 1}{Q_{n-2}(x)}$$
for all $n \geq 2$. Show that, whenever n is a positive integer, $Q_n(x)$ is equal to a polynomial with integer coefficients.

10. (Putnam 2013 B2) Let $C = \cup_{N=1}^{\infty} C_N$, where C_N denotes the set of those "cosine polynomials" of the form
$$f(x) = 1 + \sum_{n=1}^{N} a_n \cos(2\pi n x)$$
for which:

    - $f(x) \geq 0$ for all real x, and
    - $a_n = 0$ whenever n is a multiple of 3.

    Determine the maximum value of $f(0)$ as f ranges through C, and prove that this maximum is attained.

11. (Putnam 2013 A3) Suppose the real numbers $a_0, a_1, \ldots, a_n$ and x, with $0 < x < 1$, satisfy
$$\frac{a_0}{1-x} + \frac{a_1}{1-x^2} + \cdots + \frac{a_n}{1-x^{n+1}} = 0.$$
Prove that there exists a real number y with $0 < y < 1$ such that
$$a_0 + a_1 y + \cdots + a_n y^n = 0.$$

12. (Putnam 2014 A5) Let
$$P_n(x) = 1 + 2x + \cdots + nx^{n-1}.$$
Prove that the polynomials $P_j(x)$ and $P_k(x)$ are relatively prime for all positive integers j and k with $j \neq k$.

13. (Putnam 2014 B4) Show that for each positive integer n, all the roots of the polynomial
$$\sum_{k=0}^{n} 2^{k(n-k)} x^k$$
are real numbers.

14. (Putnam 2016 A6) Find the smallest constant C such that for every real polynomial $P(x)$ of degree 3 that has a root in the interval $[0, 1]$,
$$\int_0^1 |P(x)|\, dx \leq C \max_{x \in [0,1]} |P(x)|.$$