Polynomials

1) $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0, \quad a_n \neq 0$

If $a_n = 1$, it is called monic.
The set of polynomials (depending on coefficient ring) is denoted $\mathbb{C}[x]$, $\mathbb{R}[x]$, $\mathbb{Z}[x]$, etc
If $P \in \mathbb{C}[x]$, then it has n complex roots (counting the multiplicity)
If coefficients are real, then complex roots must occurs in conjugate pairs

Ex There is a polynomial P of degree 7 with integer coefficients. It is known that it is equal to ± 1 in 7 integer values. Prove that P cannot be factorized into the product of two polynomials of integer coefficents with degree ≥ 1.

□ $P(x) = Q(x)\, R(x)$
There is one polynomial of degree ≤ 3, let it Q.
$Q = \pm 1$ in 7 integer points, so at least four $+1$ or at least four -1, which is imposible since $Q \pm 1$ has degree ≤ 3 and so cannot have 4 roots ■

2) Vièteis Relations

$$x_1 + \ldots + x_n = -\frac{a_{n-1}}{a_n}$$
$$x_1 x_2 + \ldots + x_{n-1} x_n = \frac{a_{n-2}}{a_n}$$
$$\vdots$$
$$x_1 x_2 \ldots x_n = (-1)^n \frac{a_0}{a_n}$$

~~Ex Let $x+y+z=a$~~

Ex $x^4 + 3x^3 + 11x^2 + 9x + A$ has roots a, b, c, d such that $ab = cd$. Find A

□ $a+b+c+d = -3$
$-9 = abc + abd + acd + bcd = ab(c+d) + (a+b)\overset{ab}{\overset{\shortparallel}{cd}} = ab(\underbrace{a+b+c+d}_{-3}) \Rightarrow ab = 3$
$\Rightarrow A = abcd = (ab)^2 = 9$

Ex $x+y+z=0$. Prove

$$\frac{x^2+y^2+z^2}{2}\cdot\frac{x^5+y^5+z^5}{5}=\frac{x^7+y^7+z^7}{7}$$

□ consider t^3+pt+q with roots x, y, z.

Then $x^3=-px-q, \dots$

So

$$x^3+y^3+z^3=(-px-q)+(-py-q)+(-pz-q)=-p\overset{0}{(x+y+z)}-3q=-3q$$

$$x^2+y^2+z^2=\overset{0}{(x+y+z)^2}-2\overset{p}{(xy+xz+yz)}=-2p$$

$$x^4=-px^2-qx$$

$$x^4+y^4+z^4=-p(x^2+y^2+z^2)-q\overset{0}{(x+y+z)}=2p^2$$

$$x^5+y^5+z^5=-p(\underbrace{x^3+y^3+z^3}_{=-3q})-q(\underbrace{x^2+y^2+z^2}_{=-2p})=5pq$$

$$x^7+y^7+z^7=-p(\underbrace{x^5+y^5+z^5}_{=5pq})-q(\underbrace{x^4+y^4+z^4}_{=2p^2})=-7p^2q$$

So we get

$$\frac{-2p}{2}\cdot\frac{5pq}{5}=\frac{-7p^2q}{7}$$ with is true

3) Derivative: if $P=a(x-x_1)\dots(x-x_n)$

- $\frac{P'(x)}{P(x)}=\frac{1}{x-x_1}+\dots+\frac{1}{x-x_n}$
- if P has a duble root a, then $P'(a)=0$ (and if $P(a)=P'(a)=0 \Rightarrow a$ has mult. ≥ 2)
- $P'(x)$ has root between any two roots of $P(x)$ (real case)

~~Ex Find all $P(x)$ s.t. $P(x)$ is a multiple of $P''(x)$~~

~~□ $P(x)=Q(x)P''(x)$~~

~~if a is a root of P'', it should be a root of $P(x)$~~

Ex Prove that $P(x)$ is a multiple of $P'(x)$ iff $P(x)=a(x-x_0)^n$

□ $nP(x)=(x-x_0)P'(x)$ (compare top coeff.)

Let $P(x)=(x-x_0)^kQ(x)$, $Q(x_0)\neq 0$

$$\underbrace{nP(x)}_{n(x-x_0)^kQ(x)}=(x-x_0)\cdot(k(x-x_0)^{k-1}Q+(x-x_0)^kQ')\Rightarrow nQ(x)=kQ(x)+(x-x_0)Q'(x)$$

Substitute $x=x_0$ and get $nQ(x_0)=kQ(x_0)\Rightarrow n=k$ (since $Q(x_0)\neq 0$) ■

Ex Is it possible that for each a $P(x)=a$ has even number of x?

□ Take all horizontal tangent lines and lines between

• polynomials has all properties of continuous functions

Ex $Q(x) = x$ has no solution. Prove that $Q(Q(x)) - x$ has no solutio[n]

$Q(x) - x$ has no roots, so $Q(x) - x > 0$ or $Q(x) - x < 0$ for all x.
Assume
$Q(x) > x$
Then $Q(Q(x)) > Q(x) > x$.

Ex $P(x) = a_n x^n + \ldots + a_0$, $a_0 \neq 0$ has at least one real root.
Prove that one can erase all $a_i x^i$ one by one (to get a_0) in such a way that all intermediate polynomials have at least one real root

~~We do inductively if k is even erase a₀~~
~~We have $a_k x^k + \ldots + a_0$ if k is even~~

□ Let $a_0 > 0$. if n is odd, we erase one by one everything except $a_n x^n + a_0$, and then $a_n x^n$ (all the time power is odd $\Rightarrow$ ok)
~~So n is even~~ Same thing if $a_n < 0$, So n is even and $a_n > 0$. Then erase $a_n x^n$, let $Q = a_{n-1} x^{n-1} + \ldots + a_0$
if a was s.t. $P(a) = 0$, then $Q(a) = P(a) - a_n a^n < 0$, $Q(0) = a_0 > 0$
$\Rightarrow$ there is a root. ■

~~Ex~~ • Lagrange interpolation formula

$$P(x) = a_1 \frac{(x-x_2)\ldots(x-x_{n+1})}{(x_1-x_2)\ldots(x_1-x_{n+1})} + a_2 \frac{(x-x_1)(x-x_3)\ldots(x-x_{n+1})}{(x_2-x_1)(x_2-x_3)\ldots(\quad)} + \ldots$$

Ex ~~the~~ Polynomial P has rational values in all rational numbers. Prove that it has rational coefficients.