Putnam Club. Fall 2021
Number theory (October 20)

Easier problems

1. a) Show that if $a^2 + b^2 = c^2$, then $3|ab$.

   b) What is the largest k such that if $a^2 + b^2 = c^2$, then $k \mid abc$?

   c) Show that for no positive integers x and y can $2^x + 25^y$ be a perfect square.

2. The numbers 2^{2021} and 5^{2021} are written one after the other (in decimal notation). How many digits are written altogether?

3. Show that there exist 1999 consecutive numbers, each of which is divisible by the cube of some integer greater than 1.

4. Prove that if n is an integer greater than 1, then n does not divide $2^n - 1$.

5. How many primes among the positive integers, written as usual in base 10, are such that their digits are alternating 1s and 0s, beginning and ending with 1?

Putnam problems

6. (Putnam 2017) Let S be the smallest set of positive integers such that a) 2 is in S, b) n is in S whenever n^2 is in S, and c) $(n+5)^2$ is in S whenever n is in S. Which positive integers are not in S? (The set S is "smallest" in the sense that S is contained in any other such set.)

7. Show that there exists an increasing sequence $(a_n)_{n\geq 1}$ of positive integers such that for any $k \geq 0$, the sequence $k + a_n$, $n \geq 1$, contains only finitely many primes.

8. (Putnam 1995) Show that every positive integer is a sum of one or more numbers of the form $2^r 3^s$, where r and s are non-negative integers and no summand divides another.

9. (Putnam 2003) Let n be a fixed positive integer. How many ways are there to write n as a sum of positive integers,
$$n = a_1 + a_2 + \cdots a_k$$
   with k an arbitrary positive integer and $a_1 \leq a_2 \leq \cdots \leq a_k \leq a_1 + 1$? For example, with $n = 4$, there are four ways: 4, $2+2$, $1+1+2$, $1+1+1+1$.

10. (Putnam 2001) Let n be an even positive integer. Write the numbers $1, 2, \cdots, n^2$ in the squares of an $n \times n$ grid so that the kth row, from left to right, is
$$(k-1)n + 1,\ (k-1)n + 2,\ \cdots,\ (k-1)n + n.$$
    Color the squares of the grid so that half of the squares in each row and in each column are red and the other half are black (a checkerboard coloring is one possibility). Prove that for each coloring, the sum of the numbers on the red squares is equal to the sum of the numbers on the black squares.

11. (Putnam 2014) Prove that every non-zero coefficient of the Taylor series of $(1-x+x^2)e^x$ about $x = 0$ is a rational number whose numerator (in lowest terms) is either 1 or a prime number.

12. (Putnam 2014) A base 10 over-expansion of a positive integer N is an expression of the form $N = d_k 10^k + d_{k-1} 10^{k-1} + \cdots + d_0 10^0$ with $d_k \neq 0$ and $d_i \in \{0, 1, 2, \ldots, 10\}$ for all i. For instance, the integer $N = 10$ has two base 10 over-expansions: $10 = 10 \cdot 10^0$ and the usual base 10 expansion $10 = 1 \cdot 10^1 + 0 \cdot 10^0$. Which positive integers have a unique base 10 over-expansion?

13. (Putnam 2011) Let S be the set of all ordered triples (p, q, r) of prime numbers for which at least one rational number x satisfies $px^2 + qx + r = 0$. Which primes appear in seven or more elements of S?