Putnam Club. Spring 2022. Analysis II (Feb 23).

1. Let $f : [0; 1] \to \mathbb{R}$ continuous, and suppose that $f(0) = f(1)$. Show that there is a value $x \in [0; 1998/1999]$ satisfying $f(x) = f(x + 1/1999)$.
2. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous, with $f(x) \cdot f(f(x)) = 1$ for all $x \in \mathbb{R}$. If $f(1000) = 999$, find $f(500)$.
3. Prove that there are no positive numbers x and y such that $x2^y + y2^{-x} = x + y$.

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4. Assume $a > b > 0$. Prove that
$$\sqrt{ab} < \frac{a-b}{\ln a - \ln b} < \frac{a+b}{2}.$$
5. Does there exist a positive sequence a_n such that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} 1/(n^2 a_n)$ are convergent?
6. Prove that the sequence
$$a_n = \sqrt{1 + \sqrt{2 + \sqrt{3 + \ldots + \sqrt{n}}}}, \quad n \geq 1$$
is convergent
7. Assume that $f : \mathbb{R} \to \mathbb{R}$ is continuous, non-zero at least in one point, and
$$f(x + y) = f(x)f(y).$$
Prove that $f(x) = a^x$ for some $a > 0$.
8. Consider the sequence $x_1 = c$, $x_{n+1} = nx_n - 1$. Assume that there exists a finite limit of x_n, as $n \to \infty$. Find c.
9. (Putnam 1992) Let f be an infinitely differentiable real-valued function defined on the real numbers. If
$$f(\tfrac{1}{n}) = \frac{n^2}{n^2+1}, \quad n = 1, 2, 3, \ldots$$
compute the values of the derivatives $f^{(k)}(0)$, $n = 1, 2, 3, \ldots$.

   Hint: Justify that the desired derivatives must coincide with those of the function $g(x) = 1/(1 + x^2)$ by considering the function $f - g$.
10. (Putnam 2016) Let $x_0, x_1, x_2, \ldots$ be the sequence such that $x_0 = 1$ and for $n \geq 0$,
$$x_{n+1} = \ln(e^{x_n} - x_n)$$
(as usual, the function ln is the natural logarithm). Show that the infinite series
$$x_0 + x_1 + x_2 + \cdots$$
converges and find its sum.

    Hint: Use $x_n = e^{x_n} - e^{x_{n+1}}$

11. (Putnam 2014) Suppose that f is a function on the interval $[1,3]$ such that $-1 \le f(x) \le 1$ for all x and $\displaystyle\int_1^3 f(x)\,dx = 0$. How large can $\displaystyle\int_1^3 \frac{f(x)}{x}\,dx$ be?

    Hint: subtract $c\int_1^3 f(x)\,dx$ (this is equal to 0).

12. (Putnam 2016) Suppose that f is a function from $\mathbb{R} \to \mathbb{R}$ such that

$$f(x) + f\left(1 - \frac{1}{x}\right) = \arctan x$$

for all real $x \neq 0$. (As usual, $y = \arctan x$ means $-\pi/2 < y < \pi/2$ and $\tan y = x$.) Find

$$\int_0^1 f(x)\,dx.$$

Hint: iterate the function $x \to 1 - 1/x$ to find f

More challenging problems

13. Let α be a real number such that n^α is an integer for every positive integer n. Prove that α is a non-negative integer.

    Hint: Assume $\Delta f = f(x+1) - f(x)$, $\Delta^{(k)} f = \Delta(\Delta^{(k-1)} f)$. Show that for any k there exists $\theta_k \in (0,k)$ such that $\Delta^{(k)} f(x) = f^{(k)}(x + \theta_k)$, and consider $f(x) = x^\alpha$, $x = n$.

14. Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function satisfying $f(0) = 0$, $f(1) = 1$, and $f(x) \ge 0$ for all $x \in \mathbb{R}$. Show that there exist a positive integer n and a real number x such that $f^{(n)}(x) < 0$.

15. The grasshopper jumps on the segment $[0,1]$. Each of its jumps is from x to $x/\sqrt{3}$ or from x to $1 + (x-1)/\sqrt{3}$. Given any $a \in [0,1]$, prove that from any starting point the grasshopper can jump to the point within the distance smaller than 0.01 from a.