Putnam Club. Spring 2022. Complex numbers (Apr 13).

1. (warm-up) If a, b and n are positive integers, prove there exist integers x and y such that $(a^2+b^2)^n = x^2+y^2$.

2. Find a closed-form expression for $\sum_{k=0}^{n} \sin(k)$.

3. Find a closed-form expression for $\prod_{k=1}^{n-1} \sin(k\pi/n)$.

4. For positive integer n define
$$S_n = \binom{3n}{0} + \binom{3n}{3} + \ldots + \binom{3n}{3n}.$$
   (a) Find a closed-form expression for S_n.
   (b) Prove that:
$$\lim_{n\to\infty} S_n^{1/3n} = 2.$$

5. Factor $p(z) = z^5 + z + 1$.

6. Consider a regular n-gon which is inscribed in a circle with radius 1. What is the average length of its diagonal (let us agree that sides of the polygon are considered diagonals, too)? What is the limit of this expression as n goes to infinity?

7. Suppose f and g are non-constant, differentiable, real-valued functions on $\mathbb{R}$. It is known that
$$f(x+y) = f(x)f(y) - g(x)g(y);$$
$$g(x+y) = f(x)g(y) + g(x)f(y);$$
and that $f'(0) = 0$. Show that $f(x)^2 + g(x)^2 = 1$ for all x.

8. Let $p(x)$ be a polynomial with real coefficients that is non-negative for all real x. Prove that there are polynomials with real coefficients $f(x), g(x)$ such that $p(x) = f(x)^2 + g(x)^2$.

9. Let $I_m = \int_0^{2\pi} \cos(x)\cos(2x)\ldots\cos(mx)\,dx$. For which integers m, $1 \le m \le 10$, is $I_m \ne 0$?

10. Let n be a positive integer, and let $f_n(z) = n + (n-1)z + (n-2)z^2 + \cdots + z^{n-1}$. Prove that f_n has no roots in the closed unit disk $\{z \in \mathbb{C} : |z| \le 1\}$.

11. Suppose that a, b, u, v are real numbers for which $av - bu = 1$. Prove that
$$a^2 + b^2 + u^2 + v^2 + au + bv \ge \sqrt{3}.$$

12. n lights are arranged in a circle, with exactly one initially on. You are permitted to do the following: given any divisor d of n that is strictly less that n, consider the n/d lights arranged at regular intervals (every d-th light) around the circle. If all lights are in the same state, you are allowed to turn them all on (if they are off) or to turn them all off (if they are on). For which values of n is it possible to turn all the lights on by a sequence of such moves?

13. Let k be a positive integer, let $m = 2^k + 1$, and let $\alpha \ne 1$ be a complex root of $z^m - 1 = 0$. Prove that there exist polynomials $P(z)$ and $Q(z)$ with integer coefficients such that $(P(\alpha))^2 + (Q(\alpha))^2 = -1$.