STAR BARS - RETURN OF THE PIZZA

THE PIZZA CLUB

1. Glassy and Ian are in a pizza-eating competition. The maximum number of pizzas they can consume are G and I respectively. The competition has several rounds, each round going as follows:

- The winner of the previous decisive round goes first. Until the first decisive round, assume that Glassy goes first.
- The first player must eat a pizza, if possible. If the first player cannot eat a pizza, then the second player wins the round by default.
- If the first player eats a pizza, then the second player can respond by either eating a pizza (if possible) or abstaining. In the former case, the round is a draw, and in the latter case, the first player wins the round.

The competition ends when neither player can consume more pizza. Both players are aiming to maximize the number of rounds they win, and then minimize the number of rounds their opponent wins. How many rounds will Glassy and Ian each win under optimal play? (David, *Adapted from Codeforces*)

2. A signal is the data of an ordered list of positive integers $x_1, \ldots, x_n$ with cost equal to $x_1 + \cdots + x_n + n$. Determine the number of distinct signals with cost $\leqslant C$. (Haran, *Jonah and I lost a game of Singaporean Bridge*)

3. Let n be a non-negative integer. In terms of n, how many triples of non-negative integers (a, b, c) satisfy $a + 2b + 3c = 6n$? (Jonah, *Original? Classic?*)

4. Let A be a bounded, centrally symmetric, convex subset of $\mathbb{R}^2$ with area strictly greater than 4. Prove that A contains a lattice point in $\mathbb{Z}^2$ other than the origin. (Pramana, *Minkowski's Theorem*)

5. Let $d_1, d_2, \ldots, d_n$ be positive integers. Prove that there exists a positive integer N such that any multiset S with elements among the d_i's and sum divisible by N can be partitioned into multisets with sum N. (Haran, *Something Something Veronese*)

6. Let G be a weighted undirected graph with 100 vertices and 1000 edges with pairwise distinct weights. Show that there exists a trail on G consisting of 20 edges traversed in order of increasing weight. (Jonah, *Reworded from The Puzzle TOAD*)

7. Prove that for any positive integer r, there exists a positive integer N depending on r such that for any r-coloring of $\{1, 2, \ldots, N\}$, there exists a monochromatic solution to the equation $x + y = z$. (Pramana, *Schur's Theorem*)

Hints to the problems:

1. Use an inductive approach to solve small cases. Find the pattern. This is a general rule of thumb for any problem of this kind.

2. Rewrite the cost as $(x_1 + 1) + (x_2 + 1) + \cdots + (x_n + 1)$. Try some small cases.

3. There are several ways you can approach this:

- Jonah's hint: Start with $A + B + C = 6n$. How many of these have B even? You may find stars and bars helpful.
- Haran's hint: Rewrite the equation as $c + (b + c) + (a + b + c) = 6n$.
- Masochist's hint: Use functional equations.

4. Let $B = \frac{1}{2}A$. First, prove that $x + B$ cannot be pairwise disjoint sets for $x \in \mathbb{Z}^2$ using a "pigeonhole-inspired" argument.

5. Choose $N = n \cdot \mathrm{lcm}(d_1, \ldots, d_n)$. Note that $N = \mathrm{lcm}(d_1, \ldots, d_n)$ does not always work since we have the obvious counterexample:

$S = \{1, 2764080, 3004718640, 196621276005, 262161701340, 285994583280, 314594041608, 34954893512$

6. Try a greedy approach. If you stumble, you may be starting at the wrong vertex.

7. First, prove that for sufficiently large positive integers M, any r-coloring of the edges of K_M (the complete graph on M vertices) has a monochromatic triangle.

"If you have n pigeons and $n + 1$ holes, then there must exist a pigeon with at least two holes." - Sun Tzu, *The Art of War* (probably)