Putnam Club
February 5, 2025
Complex numbers

Some of these problems can be solved without complex numbers - if you found such a solution, consider whether complex number would make it easier.

Sources: [NWU]=Northwestern University Putnam preparation materials, [S]=Stanford Putnam Problem solving seminar.

1. [S] If a, b, and n are positive integers, prove there exist integers x and y such that

$$(a^2 + b^2)^n = x^2 + y^2.$$

(This is not really so exciting if you know which integers are representable as sums of two squares... What is more impressive is that x and y are polynomial in a and b.)

2. [S] Evaluate

$$\binom{2025}{0} - \binom{2025}{2} + \binom{2025}{4} - \cdots + \binom{2025}{2024}.$$

3. (continuation - but trickier) Find a simple formula for

$$\binom{2025}{0} + \binom{2025}{5} + \binom{2025}{10} - \cdots + \binom{2025}{2025}.$$

4. (And the last one in the series:) Suppose

$$f(x) = a_0 + a_1 x + a_2 x^2 + \dots$$

where $a_k \in \mathbb{R}$. Let $m \geq 0$ and $n \geq 1$ be integers. Find a method for calculating

$$S(m; n) = \sum_{k=0}^{\infty} a_{m+kn}.$$

5. [NWU] Find a closed-form expression for $\sum_{k=0}^{n} \sin(k)$.

6. [NWU] Find a closed-form expression for $\prod_{k=1}^{n-1} \sin \frac{k\pi}{n}$.

7. [NWU] n lights are arranged in a circle, with exactly one initially on. You are permitted to do the following: given any divisor d of n that is strictly less that n, consider the n/d lights arranged at regular intervals (every d-th light) around the circle. If all lights are in the same state, you are allowed to turn them all on (if they are off) or to turn them all off (if they are on). For which values of n is it possible to turn all the lights on by a sequence of such moves?

8. [S] Given a point P on the circumference of a unit circle and the vertices $A_1, A_2, \dots, A_n$ of an inscribed regular n-gon, prove that the quantity

$$|PA_1|^2 + |PA_2|^2 + \cdots + |PA_n|^2$$

is independent of P.

9. [NWU] A regular n-gon is inscribed in a unit circle. What is the product of lengths of its diagonals? (Let's agree that sides are diagonals, too.)

10. (Putnam 1991, B2): Suppose f and g are non-constant, differentiable, real-valued functions on $\mathbb{R}$. You are told that

$$f(x + y) = f(x)f(y) - g(x)g(y)$$
$$g(x + y) = f(x)g(y) + g(x)f(y),$$

and that $f'(0) = 0$. Show that $f(x)^2 + g(x)^2 = 1$ for all x.

Things to remember:

- Formulas for product of complex numbers;
- $|z_1 z_2| = |z_1| \cdot |z_2|$;
- Euler's formula: $e^{ix} = \cos(x) + i\sin(x)$ and its relatives:

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2} \qquad \sin(x) = \frac{e^{ix} - e^{-ix}}{2i}.$$

Hints:

1. $|a + bi| = \sqrt{a^2 + b^2}$.

2. What value of x would make $(1 + x)^{2025}$ useful?

3. Time to let x be a fifth root of unity.

4. Keep playing with roots of unity.

5. Use the Euler identity.

6. There is a trick to computing $\prod(1 - \zeta^i)$ where ζ is a root of unity: treat 1 as a variable (!)

7. Put the lights on a plane and start adding things up.

8. Even without complex numbers, you can just use coordinates.

9. See problem 6.

10. Consider $f(x)+g(x)i$ (or simply take $h(x) = f(x)^2+g(x)^2$ and figure out what equation it satisfies.