Putnam Club Problem Sheet - February 14

Warm-up

**Putnam 2006 A1.** Find the volume of the region of points \((x, y, z)\) such that 
\[
(x^2 + y^2 + z^2 + 8)^2 \leq 36(x^2 + y^2).
\]

**Putnam 2010 A3.** Suppose that the function \(h : \mathbb{R}^2 \to \mathbb{R}\) has continuous partial derivatives and satisfies the equation
\[
h(x, y) = a \frac{\partial h}{\partial x}(x, y) + b \frac{\partial h}{\partial y}(x, y)
\]
for some constants \(a, b\). Prove that if there is a constant \(M\) such that \(|h(x, y)| \leq M\) for all \((x, y) \in \mathbb{R}^2\), then \(h\) is identically zero.

**Putnam 2004 B3.** Determine all real numbers \(a > 0\) for which there exists a nonnegative continuous function \(f(x)\) defined on \([0, a]\) with the property that the region 
\[
R = \{(x, y); 0 \leq x \leq a, 0 \leq y \leq f(x)\}
\]
has perimeter \(k\) units and area \(k\) square units for some real number \(k\).

Harder Problems

**Putnam 2011 A5.** Let \(F : \mathbb{R}^2 \to \mathbb{R}\) and \(g : \mathbb{R} \to \mathbb{R}\) be twice continuously differentiable functions with the following properties:

- \(F(u, u) = 0\) for every \(u \in \mathbb{R}\);
- for every \(x \in \mathbb{R}\), \(g(x) > 0\) and \(x^2g(x) \leq 1\);
- for every \((u, v) \in \mathbb{R}^2\), the vector \(\nabla F(u, v)\) is either \(0\) or parallel to the vector \((g(u), -g(v))\).

Prove that there exists a constant \(C\) such that for every \(n \geq 2\) and any \(x_1, \ldots, x_{n+1} \in \mathbb{R}\), we have
\[
\min_{i \neq j} |F(x_i, x_j)| \leq \frac{C}{n}.
\]

**Putnam 2004 A6.** Suppose that \(f(x, y)\) is a continuous real-valued function on the unit square \(0 \leq x \leq 1, 0 \leq y \leq 1\). Show that
\[
\int_0^1 \left( \int_0^1 f(x, y)dx \right)^2 dy + \int_0^1 \left( \int_0^1 f(x, y)dy \right)^2 dx \leq \left( \int_0^1 \int_0^1 f(x, y)dx \ dy \right)^2 + \int_0^1 \int_0^1 [f(x, y)]^2 \ dx \ dy.
\]

**Putnam 2011 A6.** Let \(f : [0, 1]^2 \to \mathbb{R}\) be a continuous function on the closed unit square such that \(\frac{\partial f}{\partial x}\) and \(\frac{\partial f}{\partial y}\) exist and are continuous on the interior \((0, 1)^2\). Let \(a = \int_0^1 f(0, y) \ dy, b = \int_0^1 f(1, y) \ dy, c = \int_0^1 f(x, 0) \ dx, d = \int_0^1 f(x, 1) \ dx\). Prove or disprove: There must be a point \((x_0, y_0)\) in \((0, 1)^2\) such that
\[
\frac{\partial f}{\partial x}(x_0, y_0) = b - a \quad \text{and} \quad \frac{\partial f}{\partial y}(x_0, y_0) = d - c.
\]
Putnam 2018 B5. Let $f = (f_1, f_2)$ be a function from $\mathbb{R}^2$ to $\mathbb{R}^2$ with continuous partial derivatives $\frac{\partial f_i}{\partial x_j}$ that are positive everywhere. Suppose that

$$\frac{\partial f_1}{\partial x_1} \frac{\partial f_2}{\partial x_2} - \frac{1}{4} \left( \frac{\partial f_1}{\partial x_2} + \frac{\partial f_2}{\partial x_1} \right)^2 > 0$$

everywhere. Prove that $f$ is one-to-one.

Two Devilish Problems

Putnam 2012 A6. Let $f(x, y)$ be a continuous, real-valued function on $\mathbb{R}^2$. Suppose that, for every rectangular region $R$ of area 1, the double integral of $f(x, y)$ over $R$ equals 0. Must $f(x, y)$ be identically 0?

Putnam 2003 B6. Let $f(x)$ be a continuous real-valued function defined on the interval $[0, 1]$. Show that

$$\int_0^1 \int_0^1 |f(x) + f(y)| \, dx \, dy \geq \int_0^1 |f(x)| \, dx.$$