Putnam Club
September 18, 2024
Combinatorics

1. Each of the faces of a cube is colored a different color. How many of the colorings are distinct?

2. Find the sum
$$\sum_{k=1}^{n} \binom{n}{k} k^3.$$
(Consider the problem of selecting a committee, and a chairman, vice-chairman, and a secretary in this committee.)

3. How many ways are there to choose n objects from $3n+1$ objects, assuming that of these $3n+1$, n objects are indistinguishable, and the rest are all distinct?

4. How many subsets of $\{1, \ldots, n\}$ have no two successive numbers?

5. Can we arrange the numbers $1, 2, \ldots, 9$ along a circle so that the sum of two neighbors is never divisible by 3, 5, or 7?

6. (1992-B-1) Let S be a set of n distinct real numbers. Let A_S be the set of numbers that occur as averages of two distinct elements of S. For a given $n \geq 2$, what is the smallest possible number of distinct elements in A_S?

7. Is there a subset $A \subset \{1, \ldots, 3000\}$ with 2000 elements such that if $x \in A$, then $2x \notin A$?

8. Is there a polyhedron with an odd number of faces, each face having an odd number of edges?

9. Let $1 \leq r \leq n$ and consider all subsets of r elements of the set $\{1, 2, \ldots, n\}$. Each of these subsets has a minimal element. Let $F(n, r)$ denote the mean of these smallest numbers; show that
$$F(n, r) = \frac{n+1}{r+1}.$$

Harder problems.

10. (2013-B-3.) Let P be a nonempty collection of subsets of $\{1, \ldots, n\}$ such that:

(1) if $S, S' \in P$, then $S \cup S' \in P$ and $S \cap S' \in P$;

(2) if $S \in P$ and $S \neq \emptyset$, then there exists a subset $T \subset S$ such that $T \in P$ and T contains exactly one fewer element that S.

Suppose that $f : P \to \mathbb{R}$ is a function such that $f(\emptyset) = 0$ and

$$f(S \cap S') = f(S) + f(S') - f(S \cup S')$$

for all $S, S' \in P$. Must there exist real numbers $f_1, \ldots, f_n$ such that

$$f(S) = \sum_{i \in S} f_i$$

for every $S \in P$?

11. (2015-B-5.) Let P_n be the number of permutations π of $\{1, 2, \ldots, n\}$ such that $|i - j| = 1$ implies $|\pi(i) - \pi(j)| \leq 2$ for all i, j in $\{1, 2, \ldots, n\}$. Show that for $n \geq 2$, the quantity $P_{n+5} - P_{n+4} - P_{n+3} + P_n$ does not depend on n, and find its value.