1. Find the limit:

$$\lim_{x\to\infty}\left(\frac{x-1}{x+2}\right)^{x+3}.$$

2. Find the limit:

$$\lim_{x\to\infty}(x^3+x^2\sqrt[3]{1-x^3}).$$

3. Let $a_1, ..., a_n$ be a finite sequence of real numbers. Show that there exists $m, 0 \le m \le n$ such that

$$\left|\sum_{k=1}^{m} a_k - \sum_{k=m+1}^{n} a_k\right| \le \max_{1\le k\le n} |a_k|.$$

(For $m = 0$ the first sum is zero and for $m = n$ the second sum is zero.)

4. Let $\{a_k\}_{k=1}^n$ and $\{b_k\}_{k=1}^n$ be sequences of real numbers. Let $\{\hat{a}_k\}_{k=1}^n$ and $\{\hat{b}_k\}_{k=1}^n$ be non-decreasing rearrangements of these sequences and let $\{\check{a}_k\}_{k=1}^n$ and $\{\check{b}_k\}_{k=1}^n$ be non-increasing rearrangements of these sequences. Prove that

$$\sum_{k=1}^{n} \hat{a}_k\check{b}_k \le \sum_{k=1}^{n} a_k b_k \le \sum_{k=1}^{n} \hat{a}_k\hat{b}_k.$$

5. Let a_n be a sequence of positive numbers monotonically decreasing to zero. Show that if $\sum a_n = +\infty$ then $\sum \min(a_n, \frac{1}{n}) = +\infty$.

6. Let $p_1 = 2, p_2 = 3, p_3 = 5, ...$ be the sequence of all prime numbers in increasing order. Prove that

$$\left(1-\frac{1}{p_1}\right)^{-1}\left(1-\frac{1}{p_2}\right)^{-1}...\left(1-\frac{1}{p_n}\right)^{-1} > \sum_{k=1}^{p_n}\frac{1}{k}.$$

7. With p_n the same as above, prove that

$$1+\sum_{k=1}^{n}\frac{1}{p_k} > \ln\ln p_n.$$

8. Prove that the series

$$\sum_{n=1}^{\infty}\frac{1}{n^2|\sin(\pi n\sqrt{2})|}$$

converges.