THE HUNGER GAMES

PIZZA CLUB

1. Games

(1) The Pizza club has two parties: the Glass Nickels and Ian-ians. The two parties take turns buying pizza from Glass Nickel and Ian's respectively, where a turn consists of buying pizza for either 1 or 2 weeks. Both parties are keen to buy pizza from their favorite store on the 2024th week of the Pizza club. Which party has a strategy to win this privilege? What if both clubs do *not* want the 2024th week?

(2) Let k and n be integers with $1 \leqslant k \leqslant n$. Dima and Cole play a game with k pegs in a line of n holes. At the beginning of the game, the pegs occupy the k leftmost holes. A legal move consists of moving a single peg to any vacant hole that is further to the right. The players alternate moves, with Dima playing first. The game ends when the pegs are in the k rightmost holes, so whoever is next to play cannot move and therefore loses. For what values of n and k does Dima have a winning strategy? (*Putnam 2020/B2*)

(3) David and Jonah play a game with 2024 boxes of pizza. The first box has 1 pizza, the second has 2 pizzas, and so on. The players alternate turns, starting with David. On each turn, a player can perform one of the following moves:

- Eat an entire (non-empty) box of pizzas.
- Eat one pizza from each (non-empty) box.

The player who eats the last pizza loses. If both players play optimally, which player will win (other than acid reflux)? (*Original, Jonah Guse*)

2. Gam35

(1) Euclid and Dima play a game in which they take turns removing stones from a heap that initially has n stones. The number of stones removed at each turn must be one less than a prime number. Euclid goes first and the winner is the player who takes the last stone. Prove that both players have infinitely many n for which they have winning strategies. (*Putnam 2006/A2*)

(2) Dirichlet and Dima play a game. Dirichlet choses a secret positive integer n and Dima's objective is to determine Dirichlet's number. In each turn, Dirichlet takes a positive integer m as a query and reveals whether $n+m$ is prime or not. Can Dima determine n with certainty in finitely many turns? (*Asked by User Joel Newman and Answered by User Haran on MSE*)

3. GA$M_{n\times n}$ES

(1) Van and Vleck take turns filling real entries in a 2008×2008 matrix. Van goes first and the game ends when all the entries are filled. Van wins if the final matrix is invertible and Vleck wins if the matrix is singular. Which player has the winning strategy? (*Putnam 2008/A2*)

(2) Walter and Jesse are cooking matrices. Jesse picks a matrix in $\mathrm{SL}_2(\mathbb{Z}/n)$ and Walter responds with a methrix in $\mathrm{SL}_2(\mathbb{Z})$. Jesse wins if Walter's matrix does not reduce to Jesse's matrix modulo n. Who has the winning strategy? (*Suggested by Pramana Saldin*)

(3) In the 75th Annual Putnam Games, participants compete at mathematical games. Patniss and Keeta play a game in which they take turns choosing an element from $\mathrm{GL}_n(\mathbb{Z}/p)$ for a fixed positive integer n and prime p.

- A player cannot choose an element previously chosen by either player.
- A player can only choose an element commuting with all previous elements.
- A player who cannot choose an element on their term loses the game.

Patniss plays first. Which player has a winning strategy? (*Putnam 2014/B5*)

Generalization: What happens for a general finite group G? (*Asked & Answered by User Haran, Additional Remarks by Users David A Craven and Steve D on MSE*)

4. Ga-memes

(1) Batman and Joker take turns naming distinct points with integer coordinates in $[0,2] \times [0,2]$. A player wins if they name three collinear points. Does either player have a winning strategy?

(2) Superman and Bizarro simultaneously chose positive integers m and n respectively. Superman wins if $m - n \equiv 1 \bmod 3$, Bizarro wins if $m - n \equiv 2 \bmod 3$, and they play again if $m - n \equiv 0 \bmod 3$. Does either player have a winning strategy?

(3) Cole and his lads at the Pizza club are given a positive integer n.

- If n is even, then Cole will divide it by 2.
- If n is odd, then the lads will multiply it by 3 and add 1.

However, winning isn't important and everyone gets pizza if and only if n eventually becomes 1. Does any value of n pose a threat to our hunger?

"I like games" - Dima Arinkin (probably)