Fall 2019

Inequalities

Wednesday, October 2th, 2019

We now focus on understanding *inequalities for integrals.* I will list some of the well-known facts and then I will go ahead and propose some interesting problems that you might enjoy. I do think that trying to solve the problems yourself and handing the solutions to the instructor will help you gain some experience in terms of olympiads type competitions.

Facts:

- Let $f : [a, b] \to \mathbb{R}$ be a nonnegative continuous function, then

$$\int_a^b f(x)\, dx \geq 0,$$

  with equality if and only if f is identically equal to zero.

- *The Cauchy-Schwartz inequality.* Let f and g be square integrable functions. The the following inequality holds

$$\left(\int_D f(x) g(x)\, dx \right)^2 \leq \left(\int_D f(x)\, dx \right) \left(\int_D g(x)\, dx \right),$$

  where D is a "nice" domain (there are no holes and the boundary is smooth).

- *Minkowski's inequality.* It p¿1, then the following inequality holds, provided f and g are continuous functions

$$\left(\int_D |f(x) + g(x)|^p\, dx \right)^{\frac{1}{p}} \leq \left(\int_D |f(x)|^p\, dx \right)^{\frac{1}{p}} + \left(\int_D |g(x)|^p\, dx \right)^{\frac{1}{p}}.$$

  Here we also assume that D is a "nice" domain (there are no holes and the boundary is smooth).

- *Hölder's inequality.* If $p, q > 1$ are such that $1/p + 1/q = 1$ then

$$\int_D |f(x) g(x)|\, dx \leq \left(\int_D |f(x)|^p\, dx \right)^{\frac{1}{p}} \left(\int_D |g(x)|^p\, dx \right)^{\frac{1}{q}}.$$

  Same assumptions on D.

- *Generalized Hölder's inequality.* Assume that $r \in (0, \infty]$ and $p_1, \cdots p_n \in (0, \infty]$ such that

$$\sum_{k=1}^{n} \frac{1}{p_k} = \frac{1}{r},$$

(where we interpret $1/\infty$ as 0 in this equation). Then, for all measurable real- or complex-valued functions $f_1, \cdots, f_n$ defined on D,

$$\left\| \prod_{k=1}^{n} f_k \right\|_r \leq \prod_{k=1}^{n} \|f_k\|_{p_k}$$

(where we interpret any product with a factor of ∞ as ∞ if all factors are positive, but the product is 0 if any factor is 0).

In particular,

$$f_k \in L^{p_k}(\mu) \ \ \forall k \in \{1, \ldots, n\} \implies \prod_{k=1}^{n} f_k \in L^r(\mu).$$

- *Chebyshev's inequality.* Let f and g be two increasing functions on $\mathbb{R}$. Then for any real numbers $a < b$,

$$(b-a) \int_a^b f(x)g(x)\, dx \geq \left(\int_a^b f(x)\, dx \right) \left(\int_a^b g(x)\, dx \right).$$

Here are the problems we did in class. for now I will not include the solutions but if you need them, please let me know and I will add them:

1. Find all continuous functions $f : [0, 1] \to \mathbb{R}$ satisfying

$$\int_0^1 f(x)\, dx = \frac{1}{3} + \int \left(f(x^2)\right)^2\, dx.$$

2. Determine all continuous functions $f : [0, 1] \to \mathbb{R}$ that satisfy

$$\int_0^1 f(x(x - f(x)))\, dx = \frac{1}{12}.$$

3. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function such that

$$\int_0^1 f(x)\, dx = \int_0^1 x f(x)\, dx = 1.$$

Prove that

$$\int_0^1 (f(x))^2 \geq 4.$$

4. Let $f : [0, \infty) \to [0, \infty)$ be a continuous, strictly increasing function with $f(0) = 0$. Prove that

$$\int_0^a f(x)\,dx + \int_0^b f^{-1}(x)\,dx \geq ab$$

for all positive numbers a and b, with equality if and only if $b = f(a)$. Here f^{-1} denotes the inverse of the function f.

5. Let $f : [0, 1] \to [0, \infty)$ be a differentiable function with decreasing first derivative, and such that $f(0) = 0$, and $f'(0) > 0$. Prove that

$$\int_0^1 \frac{dx}{f^2(x) + 1} \leq \frac{f(1)}{f'(1)}.$$

Can equality hold?

Proposed problems for October 9th meeting

1. Prove that any continuous differentiable function $f : [a, b] \to \mathbb{R}$ for which $f(a) = 0$ satisfies the inequality

$$\int_a^b (f(x))^2\,dx \leq (b-a)^2 \int_a^b (f'(x))^2\,dx.$$

2. Let $f(x)$ be a continuous real-valued function defined on the interval $[0, 1]$. Show that

$$\int_0^1 \int_0^1 |f(x) + f(y)|\,dxdy \geq \int_0^1 |f(x)|\,dx.$$

3. Let f be a nonincreasing function on the interval $[0, 1]$. Prove that for any $\alpha \in (0, 1)$,

$$\alpha \int_0^1 f(x)\,dx \leq \int_0^\alpha f(x)\,dx.$$

4. Prove that for any positive real numbers x, y and any positive integers m, n

$$(n-1)(m-1)\left(x^{m+n} + y^{m+n}\right) + (m+n-1)(m-1)\left(x^m y^n + x^n y^m\right) \geq mn(m-1)\left(x^{m+n-1}y + y^{m+n-1}x\right).$$

5. Find the maximal values of the ratio

$$\left(\int_0^3 f(x)\,dx\right)^3 \Big/ \int_0^3 (f()x)^3\,dx,$$

as f ranges over all positive continuous functions on $[0, 1]$.

6. Find all differentiable functions $f : (0, \infty) \to (0, \infty)$ for which there is a positive real number a such that

$$f'\left(\frac{a}{x}\right) = \frac{x}{f(x)}$$

for all $x > 0$.