MORE FUNCTIONAL EQUATIONS (02/22/23)

1. Leftover problems from last time

Unless stated otherwise, 'functions' means functions of one (real) variable.

1. (Putnam'00) Let $f(x)$ be a continuous function such that $f(2x^2-1) = 2xf(x)$ for all x. Show that $f(x) = 0$ for all $-1 \le x \le 1$.

2. Find all functions $f(x)$ such that $f(\frac{x+y}{2}) + f(x) + f(y) = x + y$.

3. $f(x, y)$ is a function on the plane. It has the following property: the sum of its values in the vertices of any square is equal to zero. Show that $f = 0$.

2. And a few more (from actual competitions)

4. Find polynomials $f(x)$, $g(x)$, and $h(x)$ such that

$$|f(x)| - |g(x)| + h(x) = \begin{cases} -1, & x < -1 \\ 3x + 2, & -1 \le x \le 0 \\ -2x + 2, & x > 0. \end{cases}$$

Can you prove your solution is unique?

5. Let $f : \mathbb{R} \to \mathbb{R}$ satisfy the functional equation:

$$f(x + 1) = 1/2 + \sqrt{f(x) - f(x)^2}.$$

Prove that f is periodic, and give an example of a non-constant f satisfying the equation.

6. Show that there exists a unique function $f(x)$ defined for $x > 0$ such that $f(x) > 0$ and

$$f(f(x)) = 6x - f(x).$$

3. If you are getting bored of this type of problems

7. A function f has continuous third derivative on $\mathbb{R}$. Prove that there exists a point where

$$f(x) \cdot f'(x) \cdot f''(x) \cdot f'''(x) \ge 0.$$

4. Something to think about (or to discuss, time permitting

What is actually going on in problems 6 and 3?