Putnam Club Problem Sheet - March 22

1. Limits and Sums

Warm-up problem Show that if (a_i) is a sequence of nonnegative real numbers, then

$$\lim_{p\longrightarrow 0}\left(\frac{1}{n}\sum_{i=1}^{n} a_i^p\right)^{1/p} = \exp\left(\frac{1}{n}\sum_{i=1}^{n}\log(a_i)\right)$$

(Hint: Take logs of both sides and use the estimate $\log(1+x) \leq x$.) The continuous version of this statement, which follows by interpreting the sums as Riemann sums, is $\lim_{p\longrightarrow 0}\left(\int_0^1 |f|^p\right)^{1/p} = \exp\left(\int_0^1 \log|f|\right)$.

Problem. (2021 A2): For every positive real number x, let

$$g(x) = \lim_{r\to 0}((x+1)^{r+1} - x^{r+1})^{\frac{1}{r}}.$$

Find $\lim_{x\to\infty}\frac{g(x)}{x}$. (Hint: One approach uses the warm-up problem. Another uses L'Hopital).

Problem. (2020 A3) Let $a_0 = \pi/2$, and let $a_n = \sin(a_{n-1})$ for $n \geq 1$. Show that $\sum_{n=1}^{\infty} a_n^2$ diverges. (Hint: Compare (a_n) to a simple series you know whose sums of squares diverge)

2. Mean Value Theorem

Problem (2015 B1): Let f be a three times differentiable function (defined on $\mathbb{R}$ and real-valued) such that f has at least five distinct real zeros. Prove that $f + 6f' + 12f'' + 8f'''$ has at least two distinct real zeros. (Hint: Is it clear that f''' has two distinct real zeros? Perhaps there is a nonzero function g so that $(gf)''' = cg(f + 6f' + 12f'' + 8f''')$ for some constant c)

3. Multivariable Calculus

Problem. (2019 A4) Find a continuous function $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ whose integral over the surface of every sphere of radius 1 in $\mathbb{R}^3$ vanishes. Hint: You can find such a function $f(x,y,z)$ that only depends on z. Recall that spherical coordinates are (θ,ϕ) where $\theta \in [0,2\pi), \phi \in [0,\pi]$ and that a surface integral over the unit sphere S centered at the origin is

$$\int_S f(x,y,z)dA = \int_0^{2\pi} d\theta \int_0^{\pi} f(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)\sin\phi d\phi$$

Problem. (2018 B5) Let $f = (f_1, f_2)$ be a function from $\mathbb{R}^2$ to $\mathbb{R}^2$ with continuous partial derivatives $\frac{\partial f_i}{\partial x_j}$ that are positive everywhere. Suppose that

$$\frac{\partial f_1}{\partial x_1}\frac{\partial f_2}{\partial x_2} - \frac{1}{4}\left(\frac{\partial f_1}{\partial x_2} + \frac{\partial f_2}{\partial x_1}\right)^2 > 0$$

everywhere. Prove that f is one-to-one. (Hint: If $f(p_1) = f(p_2)$, df_p must have nontrivial kernel for some p on the line from p_1 to p_2, i.e. there is some nonzero vector v so that $v^T df_p v = 0$.)

Problem. (2021 B3): Let $h(x, y)$ be a real-valued function that is twice continuously differentiable throughout $\mathbb{R}^2$, and define

$$\rho(x, y) = yh_x - xh_y.$$

Prove: For any positive constants d and r with $d > r$, there is a circle $\mathcal{S}$ of radius r whose center is a distance d away from the origin such that the integral of ρ over the interior of $\mathcal{S}$ is zero. (Hint: The gradient of h is (h_x, h_y). At a point (x, y) of distance $r_0 > 0$ from the origin, $(y, -x)$ is the tangent vector to the circle of radius r_0. So $\rho(x, y)$ is essentially the derivative of h restricted to the circle of radius r_0 at the point (x, y).)

4. Expected Value

Problem. (2020 B3) Let $x_0 = 1$, and let δ be some constant satisfying $0 < \delta < 1$. Iteratively, for $n = 0, 1, 2, \ldots$, a point x_{n+1} is chosen uniformly from the interval $[0, x_n]$. Let Z be the smallest value of n for which $x_n < \delta$. Find the expected value of Z, as a function of δ. (Hint: If $E(\delta)$ is the expected value, then set up and solve a linear first order ODE that E satisfies)

Problem. (2022 A4): Suppose that $X_1, X_2, \ldots$ are real numbers between 0 and 1 that are chosen independently and uniformly at random. Let $S = \sum_{i=1}^{k} X_i/2^i$, where k is the least positive integer such that $X_k < X_{k+1}$, or $k = \infty$ if there is no such integer. Find the expected value of S. (Hint: Let A_i be the event that $X_i < X_j$ for all $j < i$ and let B_i be the event that (X_j) is nondecreasing for $1 \leq j \leq i$, which are independent events. Find $E\left[\frac{X_i}{2^i}\middle|A_i \cap B_i\right]$).