**Generating functions and telescoping series.**
**03/30/22**

**Generating functions:**
Encode the sequence $a_0, a_1, \dots$ via

$$f(x) = \sum_{k=0}^{\infty} a_k x^k.$$

**Telescoping series:**

$$(a_1 - a_0) + (a_2 - a_1) + \cdots + (a_n - a_{n-1}) = a_n - a_0.$$

(Source: NWU Putnam preparation.)

**1.** Show that

$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}.$$

**2.** (Follow-up) Find a closed formula for

$$\sum_{k=0}^{n} k^2\binom{n}{k}.$$

**3.** Let $F_n$ be the Fibonacci sequence: $F_0 = 0$, $F_1 = 1$, $F_k = F_{k-1} + F_{k-2}$.
Find

$$\sum_{k=0}^{\infty} \frac{F_k}{2^k}.$$

**4.** (Follow-up) Find

$$\sum_{k=0}^{\infty} \frac{k \cdot F_k}{2^k}.$$

**5.** How many different sequences are there that satisfy all of the following conditions:

- The terms of the sequences are the digits 0–9.
- The length of the sequences is 6 (e.g. 061030).
- Repetitions are allowed.
- The sum of the digits is exactly 10 (e.g. 111322).

**6.** Prove that

$$\frac{1}{1+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \cdots + \frac{1}{\sqrt{99}+\sqrt{100}} = 9.$$

**7.** Let $N$ be a positive integer and let $S_N$ be the sum

$$S_N = \frac{1}{2}\sum_{k=1}^{N^2} \frac{1}{\sqrt{k}}.$$

Find $[S_N]$, where $[x]$ is the largest integer less than or equal to $x$.

**8.** Find a closed form for

$$\sum_{k=1}^{n} k \cdot k!$$

**9.** (Putnam 1984) Express

$$\sum_{k=1}^{\infty} \frac{6^k}{(3^{k+1} - 2^{k+1})(3^k - 2^k)}$$

as a rational number.

**10.** (Putnam 1977) Evaluate the infinite product

$$\prod_{n=2}^{\infty} \frac{n^3 - 1}{n^3 + 1}.$$

**11.** Evaluate the infinite series:

$$\sum_{n=0}^{\infty} \frac{n}{n^4 + n^2 + 1}.$$