Fall 2017

Analysis problems and a mix of random questions (mainly from probability theory)

Wednesday, October 18th, 2017

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- 1. Let $f : [0, \infty) \to \mathbb{R}$ be a twice-differentiable function satisfying $f(0) \ge 0$ and f'(x) > f(x) for all x > 0. Prove that f(x) > 0 for all x > 0.
- 2. Let f be a real-valued continuous nonnegative function on [0, 1] such that

$$f(t)^2 \le 1 + 2 \int_0^t f(s) \, ds$$
, for all $t \in [0, 1]$.

Show that $f(t) \leq 1 + t$ for every $t \in [0, 1]$.

- 3. Does there exist a continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$ satisfying f(x) > 0 and f'(x) = f(f(x)) for every $x \in \mathbb{R}$?
- 4. Determine all *n*th-degree polynomials P(x), with real zeros, for which the equality

$$\sum_{i=1}^{n} \frac{1}{P(x) - x_i} = \frac{n^2}{xP'(x)}$$

holds for all nonzero real numbers x for which $P'(x) \neq 0$, where $x_i, i = 1, 2, \dots, n$, are the zeros of P(x).

5. Let C be the class of all real-valued continuously differentiable functions f on the interval [0,1] with f(0) = 0 and f(1) = 1. Determine

$$u = \inf_{f \in C} \int_0^1 |f'(x) - f(x)| \, dx.$$

6. Prove that

$$\begin{bmatrix} 2k\\k \end{bmatrix} = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \left(2\sin\theta\right)^{2k} \, d\theta.$$

7. Show that for any positive integer n, the number

$$S_n = \begin{bmatrix} 2n+1\\ 0 \end{bmatrix} \cdot 2^{2n} + \begin{bmatrix} 2n+1\\ 2 \end{bmatrix} \cdot 2^{2n-2} \cdot 3 + \dots + \begin{bmatrix} 2n+1\\ 2n \end{bmatrix} \cdot 3^n$$

is the sum of two consecutive perfect squares.

- 8. A coin is tossed n times. What is the probability that two heads will turn up in succession somewhere in the sequence?
- 9. What is the probability that three points selected at random on a circle lie on a semicircle?
- 10. Let $n \ge 4$ be given, and suppose that the points P_1, P_2, \ldots, P_n are randomly chosen on a circle. Consider the convex *n*-gon whose vertices are these points. what is the probability that at least one of the vertex angles of this polygon is acute?